

Stable reconstructions in Hilbert spaces and the resolution of the Gibbs phenomenon

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Abstract

We introduce a method to reconstruct an element of a Hilbert space in terms of an arbitrary finite collection of linearly independent reconstruction vectors, given a finite number of its samples with respect to any Riesz basis. As we establish, provided the dimension of the reconstruction space is chosen suitably in relation to the number of samples, this procedure can be numerically implemented in a stable manner. Moreover, the accuracy of the resulting approximation is completely determined by the choice of reconstruction basis, meaning that the reconstruction vectors can be tailored to the particular problem at hand.

An important example of this approach is the accurate recovery of a piecewise analytic function from its first few Fourier coefficients. Whilst the standard Fourier projection suffers from the Gibbs phenomenon, by reconstructing in a piecewise polynomial basis, we obtain an approximation with root exponential accuracy in terms of the number of Fourier samples and exponential accuracy in terms of the degree of the reconstruction function. Numerical examples illustrate the advantage of this approach over other existing methods.

1 Introduction

Suppose that H is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. In this paper, we consider following problem: given the first m samples $\{\langle f, \psi_j \rangle\}_{j=1}^m$ of an element $f \in H$ with respect to some Riesz basis $\{\psi_j\}_{j=1}^\infty$ of H (the *sampling* basis), reconstruct f to high accuracy. Not only does such a problem lie at the heart of modern sampling theory [21, 47], it also occurs in a myriad of applications, including image processing (in particular, Magnetic Resonance Imaging), and the numerical solution of hyperbolic partial differential equations (PDEs).

In practice, straightforward reconstruction of f may be achieved via orthogonal projection with respect to the sampling basis. Indeed, for an arbitrary $f \in H$, this is the best possible strategy. However, in many important circumstances, this approximation converges only slowly in m , when

measured in the norm on H , or not at all, if a stronger norm (for example, the uniform norm) is considered.

A prominent instance of this problem is the recovery of a function $f : \mathbb{T} \rightarrow \mathbb{R}$ from its first m Fourier coefficients (here $\mathbb{T} = [-1, 1)$ is the unit torus). In this instance, $H = L^2(\mathbb{T})$ is the space of all 2-periodic, square-integrable functions of one variable. Provided f is analytic, it is well-known that its Fourier series (the orthogonal projection with respect to the Fourier basis) converges exponentially fast. However, whenever f has a jump discontinuity, its Fourier series suffers from the well-known Gibbs phenomenon [41]. Whilst convergence occurs in the L^2 norm, uniform convergence is lacking, and the approximation is polluted by characteristic $\mathcal{O}(1)$ oscillations near the discontinuity. Moreover, the rate of convergence is also slow: only $\mathcal{O}(m^{-\frac{1}{2}})$ when measured in the L^2 norm, and $\mathcal{O}(m^{-1})$ pointwise away from the discontinuity. Needless to say, the Gibbs phenomenon is a significant blight of many practical applications of Fourier series [35]. It is a testament to its importance that the design of effective techniques for its removal remains an active area of inquiry [30, 45].

Returning to the general form of the problem, let us now suppose that some additional information is known about the function f . For example, f may be sparse in a particular basis (e.g. a polynomial of low degree) or may possess certain regularity. In particular, in the Fourier instance, we may know that f is piecewise analytic with jump discontinuities at known locations in \mathbb{T} . In this circumstance, it seems plausible that a better approximation to f can be obtained by expanding in a different basis (e.g. a piecewise polynomial basis). To this end, let us introduce the so-called *reconstruction* space (of dimension n) and seek to approximate f by an element $f_{n,m}$ consisting of n linearly independent elements of this space.

As we will show in due course, provided reconstruction is carried out in a certain manner, a suitable approximation $f_{n,m}$ can always be found. Essential to this approach is that m (the number of samples) is chosen sufficiently large in comparison to n (equivalently, n is chosen sufficiently small in comparison to m). However, provided this is the case, the approximation $f_{n,m}$ inherits the principal features of the reconstruction space. In particular, $f_{n,m}$ is quasi-optimal (or, under certain conditions, asymptotically optimal), in sense that the error $\|f - f_{n,m}\|$ can be bounded by a constant multiple of $\|f - \mathcal{Q}_n f\|$, where $\mathcal{Q}_n f$ is the orthogonal projection onto the reconstruction space (in other words, the best approximation to f from this space). Moreover, from a practical standpoint, this method can be implemented by solving a linear least squares problem. Whenever the reconstruction vectors are suitably chosen (e.g. if they form a Riesz basis), the corresponding linear system is well-conditioned and the least squares problem can be solved in $\mathcal{O}(mn)$ operations by standard iterative techniques.

Consider once more the example of Fourier series. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be an analytic function with jump discontinuity at $x = -1$ (equivalently, an analytic, nonperiodic function). As mentioned, the Fourier series of f lacks uniform convergence. However, since f is analytic, it makes sense to seek to reconstruct f in a system of polynomials. It is well-known that the best n^{th} degree polynomial approximation of an analytic function converges exponentially fast in n [9]. As we shall prove, with $n = \mathcal{O}(\sqrt{m})$, we obtain a quasi-optimal polynomial approximation $f_{n,m}$ to f from only its first m Fourier coefficients, regardless of the particular family of polynomials used. This results in exponential convergence of $f_{n,m}$ to f in n (the polynomial degree), or root exponential convergence in m (the number of Fourier samples). Moreover, whenever Legendre polynomials are used, the approximation is computable in a stable manner in only $\mathcal{O}(mn)$ operations. The use of, for example, Chebyshev polynomials results in a method with $\mathcal{O}(n)$ condition number, that can be implemented in $\mathcal{O}(mn^{\frac{3}{2}})$ operations. Furthermore, whilst retaining the aforementioned features, this procedure can be easily generalised to recovery of a piecewise analytic function of one variable (using piecewise polynomial bases), and to the case of multivariate functions defined in tensor-product regions.

There are a number of existing algorithms for the removal of the Gibbs phenomenon from Fourier series. One of the most well-known, which also provides a polynomial approximant, is the Gegenbauer reconstruction technique [29, 30, 31]. As we discuss further in Section 3, the method developed

in this paper has a number of key advantages over this device. Numerical results also indicate its superior performance for a number of test problems.

The method developed in this paper was previously introduced by the authors in [3] within the context of abstract sampling theory. Whilst this problem, in this abstract form, has been extensively studied in the last couple of decades (in particular, by Eldar et al [20, 21], see also [47]), to the best of our knowledge this particular method does not appear in any existing literature. For a more detailed discussion of the relation of this approach to existing schemes we refer the reader to [3]. Conversely, in this paper, after presenting the general version of the method in abstract terms, we will focus primarily on its application to the Fourier coefficient reconstruction problem. On this topic, a similar approach, but only dealing with reconstructions in Legendre polynomials from Fourier samples of analytic functions, was discussed in [32]. This can be viewed as a special case of our general framework. Furthermore, by examining this example as part of this framework, we are able to extend and improve the work of [32] in the following ways: (i) we derive a procedure allowing for reconstructions in any polynomial basis, not just Legendre polynomials, (ii) we extend this approach to reconstructions of piecewise smooth functions using (arbitrary) piecewise polynomial bases, (iii) we generalise this work to smooth functions of arbitrary numbers of variables and (iv) we obtain improved estimates for both the error and the necessary scaling $n = \mathcal{O}(\sqrt{m})$ required for implementation.

Aside from yielding these improvements, a great benefit of the general framework presented in this paper is that it is immediately applicable to a whole host of other reconstruction problems. To illustrate this generality, in the final part of this paper we consider its application to the accurate reconstruction of a piecewise analytic function from its orthogonal polynomial expansion coefficients. Such a problem is typical of that occurring in the application of polynomial spectral methods to hyperbolic PDEs [27], where the shock formation inhibits fast convergence of the polynomial approximation. As we highlight, this issue can be overcome in a completely stable fashion by reconstructing in a piecewise polynomial basis.

The outline of the remainder of this paper is as follows. In Section 2 we introduce the reconstruction procedure and establish both stability and error estimates. Section 3 is devoted to (piecewise) polynomial reconstructions from Fourier samples. In Section 4 we consider reconstructions from tensor-product spaces, and in Section 5 we discuss other reconstruction problems. Finally, in Section 6 we present open problems and challenges.

2 General theory of reconstruction

In this section, we describe the reconstruction procedure in its full generality. To this end, suppose that $\{\psi_j\}_{j=1}^\infty$ is a Riesz basis (the sampling basis) for a separable Hilbert space H over the field \mathbb{C} . Let $\langle \cdot, \cdot \rangle$ be the inner product on H , with associated norm $\|\cdot\|$. Recall that, by definition, $\text{span}\{\psi_1, \psi_2, \dots\}$ is dense in H and

$$c_1 \sum_{j=1}^\infty |\alpha_j|^2 \leq \left\| \sum_{j=1}^\infty \alpha_j \psi_j \right\|^2 \leq c_2 \sum_{j=1}^\infty |\alpha_j|^2, \quad \forall \alpha = \{\alpha_1, \alpha_2, \dots\} \in \ell^2(\mathbb{N}), \quad (2.1)$$

for positive constants c_1, c_2 . Equivalently, $\psi_j = \mathcal{U}(\Psi_j)$, where $\{\Psi_j\}_{j=1}^\infty$ is an orthonormal basis for H and $\mathcal{U} : H \rightarrow H$ is a bounded, bijective operator. Using this definition, it is easy to deduce that $\{\psi_j\}_{j=1}^\infty$ also satisfies the frame property

$$d_1 \|f\|^2 \leq \sum_{j=1}^\infty |\langle f, \psi_j \rangle|^2 \leq d_2 \|f\|^2, \quad \forall f \in H, \quad (2.2)$$

for $d_1, d_2 > 0$, where the largest possible value for d_2 is $\|\mathcal{U}\|_{H \rightarrow H}^2$ and the smallest possible value for d_1 is $\|\mathcal{U}^{-1}\|_{H \rightarrow H}^{-2}$ [17].

Suppose now that the first m coefficients of an element $f \in H$ with respect to the sampling basis are given:

$$\hat{f}_j = \langle f, \psi_j \rangle, \quad j = 1, \dots, m. \quad (2.3)$$

Set $S_m = \text{span}\{\psi_1, \dots, \psi_m\}$ and let $\mathcal{P}_m : H \rightarrow S_m$ be the mapping

$$f \mapsto \mathcal{P}_m f = \sum_{j=1}^m \langle f, \psi_j \rangle \psi_j. \quad (2.4)$$

We now seek to reconstruct f in a different basis. To this end, suppose that $\{\phi_1, \dots, \phi_n\}$ are linearly independent reconstruction vectors and define $T_n = \text{span}\{\phi_1, \dots, \phi_n\}$. Let $\mathcal{Q}_n : H \rightarrow T_n$ be the orthogonal projection onto T_n . Direct computation of $\mathcal{Q}_n f$, the best approximation to f from T_n , is not possible, since the coefficients $\langle f, \phi_j \rangle$ are unknown. Instead, we seek to use the values (2.3) to compute an approximation $f_{n,m} \in T_n$ that is *quasi-optimal*, i.e. $\|f - \mathcal{Q}_n f\| \leq \|f - f_{n,m}\| \leq C\|f - \mathcal{Q}_n f\|$ for some constant $C > 0$ independent of f , n and m . To do this, we introduce the sesquilinear form $a_m : H \times H \rightarrow \mathbb{C}$, given by

$$a_m(g, h) = \langle \mathcal{P}_m g, h \rangle, \quad \forall g, h \in H. \quad (2.5)$$

Note that, since

$$\langle \mathcal{P}_m g, h \rangle = \sum_{j=1}^m \langle f, \psi_j \rangle \overline{\langle g, \psi_j \rangle} = \overline{a_m(g, f)}, \quad \forall f, g \in H,$$

a_m is a Hermitian form on $H \times H$ (here \bar{z} is the complex conjugate of $z \in \mathbb{C}$). With this to hand, we now define $f_{n,m}$ by the following condition

$$a_m(f_{n,m}, \phi) = a_m(f, \phi), \quad \forall \phi \in T_n. \quad (2.6)$$

Upon setting $\phi = \phi_j$, $j = 1, \dots, n$, this becomes an $n \times n$ linear system of equations for the coefficients $\alpha_1, \dots, \alpha_n$ of $f_{n,m} = \sum_{j=1}^n \alpha_j \phi_j$. We shall defer a discussion of the computation of this approximation to Section 2.3: in the remainder of this section we consider the analysis of $f_{n,m}$.

Before proving the main theorem regarding (2.6), let us first give an explanation as to why this approach works. As mentioned, key to this technique is that the parameter m is sufficiently large in comparison to n . To this end, let n be fixed and suppose that $m \rightarrow \infty$. Due to (2.1), the mappings \mathcal{P}_m converge strongly to a bounded, linear operator \mathcal{P} , given by

$$\mathcal{P}f = \sum_{j=1}^{\infty} \langle f, \psi_j \rangle \psi_j, \quad \forall f \in H. \quad (2.7)$$

Hence, for large m , the equations (2.6) defining $f_{n,m}$ resemble the equations

$$a(\tilde{f}_n, \phi) = a(f, \phi), \quad \forall \phi \in T_n, \quad (2.8)$$

where $a : H \times H \rightarrow \mathbb{C}$ is the Hermitian form $a(f, g) = \langle \mathcal{P}f, g \rangle$. Thus, it is reasonable to expect that $f_{n,m} \rightarrow \tilde{f}_n$ as $m \rightarrow \infty$, provided such a function \tilde{f}_n exists. However,

Theorem 2.1. *For all $n \in \mathbb{N}$, the function \tilde{f}_n exists and is unique. Moreover,*

$$\|f - \tilde{f}_n\| \leq \frac{d_2}{d_1} \|f - \mathcal{Q}_n f\|, \quad (2.9)$$

where d_1 and d_2 arise from (2.2).

This theorem can be established with a straightforward application of the Lax–Milgram theorem and its counterpart, Céa’s lemma [25]. Indeed, due to (2.2) and (2.7),

$$d_1 \|g\|^2 \leq a(g, g) = \sum_{j=1}^{\infty} |\langle g, \psi_j \rangle|^2 \leq d_2 \|g\|^2, \quad \forall g \in H. \quad (2.10)$$

Hence the form $a(\cdot, \cdot)$ defines an equivalent inner product on H . Nonetheless, we shall present a self-contained proof, since similar techniques will be used subsequently.

Proof. Let $\mathcal{U} : T_n \rightarrow \mathbb{C}^n$ be the linear mapping $g \mapsto \{\langle \mathcal{P}g, \phi_j \rangle\}_{j=1}^n$. To prove existence and uniqueness of \tilde{f}_n it suffices to show that \mathcal{U} is invertible, upon which it follows that $\tilde{f}_n = \mathcal{U}^{-1}\{\langle \mathcal{P}f, \phi_j \rangle\}_{j=1}^n$. Suppose that $\mathcal{U}g = 0$. Then, by definition, $\langle \mathcal{P}g, \phi_j \rangle = 0$ for $j = 1, \dots, n$. Using linearity, we deduce that $\langle \mathcal{P}g, g \rangle = 0$. Now, it follows from (2.10) that $0 = \langle \mathcal{P}g, g \rangle \geq d_1 \|g\|^2$, giving $g = 0$. Hence, \mathcal{U} is invertible and \tilde{f}_n exists and is unique.

Now consider the error estimate (2.9). Using (2.2) once more, we obtain

$$\|f - \tilde{f}_n\|^2 \leq \frac{1}{d_1} \langle \mathcal{P}(f - \tilde{f}_n), f - \tilde{f}_n \rangle.$$

By definition of \tilde{f}_n , $\langle \mathcal{P}(f - \tilde{f}_n), \phi \rangle = 0$, $\forall \phi \in T_n$. In particular, setting $\phi = \tilde{f}_n - \mathcal{Q}_n f$, yields

$$\|f - \tilde{f}_n\|^2 \leq \frac{1}{d_1} \langle \mathcal{P}(f - \tilde{f}_n), f - \mathcal{Q}_n f \rangle.$$

Since $a(\cdot, \cdot)$ forms an equivalent inner product on H , an application of the Cauchy–Schwarz inequality gives

$$\|f - \tilde{f}_n\|^2 \leq \frac{1}{d_1} \left[\langle \mathcal{P}(f - \tilde{f}_n), f - \tilde{f}_n \rangle \langle \mathcal{P}(f - \mathcal{Q}_n f), f - \mathcal{Q}_n f \rangle \right]^{\frac{1}{2}} \leq \frac{d_2}{d_1} \|f - \tilde{f}_n\| \|f - \mathcal{Q}_n f\|,$$

as required. \square

This theorem establishes existence and quasi-optimality of $\tilde{f}_n \approx f_{n,m}$, thereby giving an intuitive argument for the success of this method. We now wish to fully confirm this observation. To this end, let

$$C_{n,m} = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} \sum_{j=1}^m |\langle \phi, \psi_j \rangle|^2 = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} \langle \mathcal{P}_m \phi, \phi \rangle = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} a_m(\phi, \phi). \quad (2.11)$$

The quantity $C_{n,m}$ plays a fundamental role in this paper. Note that

Lemma 2.2. *For all $n, m \in \mathbb{N}$, $0 \leq C_{n,m} \leq d_2$. Moreover, for each n , $C_{n,m} \rightarrow C_n^* \geq d_1$ as $m \rightarrow \infty$, where*

$$C_n^* = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} \sum_{j=1}^{\infty} |\langle \phi, \psi_j \rangle|^2 = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} \langle \mathcal{P} \phi, \phi \rangle = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} a(\phi, \phi),$$

and d_1 is defined in (2.2).

Proof. Consider first the quantity

$$\epsilon_{n,m} = \sup_{\substack{\phi \in T_n \\ \|\phi\|=1}} \sum_{j>m} |\langle \phi, \psi_j \rangle|^2 = \sup_{\substack{\phi \in T_n \\ \|\phi\|=1}} \langle \mathcal{P} \phi - \mathcal{P}_m \phi, \phi \rangle. \quad (2.12)$$

Due to (2.2), the infinite sum is finite for any fixed ϕ , and tends to zero as $m \rightarrow \infty$. Now let $\{\Phi_j\}_{j=1}^n$ be an orthonormal basis for T_n and set $\phi = \sum_{j=1}^n \alpha_j \Phi_j$. Two applications of the Cauchy–Schwarz inequality gives

$$\sum_{j>m} |\langle \phi, \psi_j \rangle|^2 \leq \|\phi\|^2 \sum_{k=1}^n \sum_{j>m} |\langle \Phi_k, \psi_j \rangle|^2.$$

Hence $\epsilon_{n,m} \leq \sum_{k=1}^n \sum_{j>m} |\langle \Phi_k, \psi_j \rangle|^2$, and we deduce that $\epsilon_{n,m}$ is both finite and $\epsilon_{n,m} \rightarrow 0$ as $m \rightarrow \infty$. Noticing that $|C_{n,m} - C_n^*| \leq \epsilon_{n,m}$, $\forall n, m \in \mathbb{N}$, gives the first part of the proof. For the second, we merely use (2.2). \square

Aside from $C_{n,m}$, we also define the quantity $D_{n,m}$ by

$$D_{n,m} = \sup_{\substack{f \in T_n^\perp \\ \|f\|=1}} \sup_{\substack{g \in T_n \\ \|g\|=1}} |\langle \mathcal{P}_m f, g \rangle|. \quad (2.13)$$

For this, we have the following lemma:

Lemma 2.3. *For all $m, n \in \mathbb{N}$, $0 \leq D_{n,m} \leq d_2$. Moreover, suppose that \mathcal{P} is such that $\mathcal{P}(T_n) \subseteq T_n$ (for example, when $\mathcal{P} = \mathcal{I}$ is the identity), then $D_{n,m}^2 \leq c_2 \epsilon_{n,m}$, where $\epsilon_{n,m}$ is as in (2.12). In particular, for fixed n , $D_{n,m} \rightarrow 0$ as $m \rightarrow \infty$.*

Proof. Let $f, g \in H$. By definition,

$$\langle \mathcal{P}_m f, g \rangle = \sum_{j=1}^m \langle f, \psi_j \rangle \overline{\langle g, \psi_j \rangle} \leq \left[\sum_{j=1}^m |\langle f, \psi_j \rangle|^2 \right]^{\frac{1}{2}} \left[\sum_{j=1}^m |\langle g, \psi_j \rangle|^2 \right]^{\frac{1}{2}}. \quad (2.14)$$

Hence (2.2) now gives the first result. Now suppose that $f \in T_n^\perp$ and $\mathcal{P}(T_n) \subseteq T_n$. Since \mathcal{P}_m is self-adjoint, we have $\langle \mathcal{P}_m f, g \rangle = \langle f, \mathcal{P}_m g \rangle = \langle f, \mathcal{P}_m g - \mathcal{P}g \rangle$. Here the second equality is due to the fact that $f \perp T_n$ and $\mathcal{P}g \in T_n$ for $g \in T_n$. By the Cauchy–Schwarz inequality, we obtain

$$D_{n,m} \leq \sup_{\substack{g \in T_n \\ \|g\|=1}} \|\mathcal{P}g - \mathcal{P}_m g\|.$$

For $g \in T_n$, we note from (2.1) that

$$\|\mathcal{P}g - \mathcal{P}_m g\|^2 \leq c_2 \sum_{j>m} |\langle g, \psi_j \rangle|^2 = c_2 \langle \mathcal{P}g - \mathcal{P}_m g, g \rangle \leq c_2 \epsilon_{n,m} \|g\|^2,$$

where the final equality follows from the definition of $\epsilon_{n,m}$. \square

At this moment, we mention the following important fact. The constants $C_{n,m}$, $D_{n,m}$ and C_n^* , as well as the approximation $f_{n,m}$, are determined only by the space T_n , not by the choice of reconstruction vectors ϕ_1, \dots, ϕ_n themselves. As we shall discuss later, the choice of reconstruction vectors only affects the stability of the scheme.

This observation aside, we are now able to state the main theorem of this section:

Theorem 2.4. *For every $n \in \mathbb{N}$ there exists an m_0 such that the approximation $f_{n,m}$, defined by (2.6), exists and is unique for all $m \geq m_0$, and satisfies the stability estimate $\|f_{n,m}\| \leq d_2 C_{n,m}^{-1} \|f\|$. Furthermore,*

$$\|f - \mathcal{Q}_n f\| \leq \|f - f_{n,m}\| \leq K_{n,m} \|f - \mathcal{Q}_n f\|, \quad K_{n,m} = \sqrt{1 + D_{n,m}^2 C_{n,m}^{-2}}. \quad (2.15)$$

Specifically, the parameter m_0 is the least value of m such that $C_{n,m} > 0$.

To prove this theorem, we first recall that a Hermitian form $a : H \times H \rightarrow \mathbb{R}$ is said to be *continuous* if, for some constant $\gamma > 0$, $|a(f, g)| \leq \gamma \|f\| \|g\|$ for all $f, g \in H$. Moreover, a is *coercive*, provided $a(f, f) \geq \omega \|f\|^2$, $\forall f \in H$, for $\omega > 0$ constant [25]. We now require the following lemma:

Lemma 2.5. *Suppose that $a_m : H \times H \rightarrow \mathbb{R}$ is the sesquilinear form $a_m(f, g) = \langle \mathcal{P}_m f, g \rangle$. Then, a_m is continuous with constant $\gamma \leq d_2$. Moreover, for every $n \in \mathbb{N}$ there exists an m_0 such that the restriction of a_m to $T_n \times T_n$ is coercive for all $m \geq m_0$. Specifically, if $C_{n,m}$ is given by (2.11), then m_0 is the least value of m such that $C_{n,m} > 0$, and, for all $m \geq m_0$, $a_m(f, f) \geq C_{n,m} \|f\|^2$, $\forall f \in T_n$. Finally, for all $f \in H$ and $g \in T_n$, we have $a_m(f - \mathcal{Q}_n f, g) \leq D_{n,m} \|f - \mathcal{Q}_n f\| \|g\|$.*

Proof. Continuity follows immediately from (2.14). For the second and final results, we merely use the definitions (2.11) and (2.13) of $C_{n,m}$ and $D_{n,m}$ respectively. \square

Proof of Theorem 2.4. To establish existence and uniqueness, it suffices to prove that the linear operator $\mathcal{U} : T_n \rightarrow \mathbb{C}^n$, $g \mapsto \{\langle \mathcal{P}_m g, \phi_j \rangle\}_{j=1}^n$ is invertible. Suppose that $g \in T_n$ with $\mathcal{U}g = 0$. By definition, we have $\langle \mathcal{P}_m g, \phi_j \rangle = 0$ for $j = 1, \dots, n$. Using linearity, it follows that $\langle \mathcal{P}_m g, g \rangle = 0$. Lemma 2.5 now gives $0 \leq C_{n,m} \|g\|^2 \leq 0$. Hence $g = 0$, and therefore \mathcal{U} has trivial kernel.

Stability of $f_{n,m}$ is easily established from the continuity and coercivity conditions. Setting $\phi = f_{n,m}$ in (2.6) gives

$$C_{n,m} \|f_{n,m}\|^2 \leq a_m(f_{n,m}, f_{n,m}) = a_m(f, f_{n,m}) \leq d_2 \|f\| \|f_{n,m}\|,$$

as required. Now consider the error estimate (2.15). Suppose that we define $e_{n,m} = f_{n,m} - \mathcal{Q}_n f \in T_n$. Then, by definition of $f_{n,m}$, we have $a_m(e_{n,m}, \phi) = a_m(f - \mathcal{Q}_n f, \phi)$, $\forall \phi \in T_n$. In particular, setting $\phi = e_{n,m}$, we obtain

$$\|e_{n,m}\| \leq C_{n,m}^{-1} D_{n,m} \|f - \mathcal{Q}_n f\|. \quad (2.16)$$

Since $\mathcal{Q}_n f$ is the orthogonal projection onto T_n , we have $\|f - f_{n,m}\|^2 = \|e_{n,m}\|^2 + \|f - \mathcal{Q}_n f\|^2$, which gives the full result. \square

When a_m is shown to be continuous and coercive, it may be tempting to seek to apply the Lax–Milgram theorem and Céa’s lemma to obtain Theorem 2.4. However, the Hermitian form a_m , when considered as a mapping $H \times H \rightarrow \mathbb{C}$, will not, in general, be coercive. This is readily seen from the definition of $C_{n,m}$. The finite-dimensional operator $\mathcal{P}_m|_{T_n}$ converges uniformly to $\mathcal{P}|_{T_n}$, whereas its infinite-dimensional counterpart \mathcal{P}_m typically does not (for example, when \mathcal{P}_m is the Fourier projection operator and $H = L^2(\mathbb{T})$). Hence, a_m only becomes coercive when restricted to $T_n \times T_n$, and these standard results do not automatically apply.

Though Theorem 2.4 establishes an estimate for the error $f - f_{n,m}$ measured in the natural norm on H , it is also useful to derive a result valid for any other norm defined on a suitable subspace of H (for example, this may be the uniform norm on \mathbb{T} in the case of Fourier series). To this end, let $\|\cdot\|$ be such a norm and define $G = \{g \in H : \|g\| < \infty\}$. We have

Corollary 2.6. *Suppose that $f \in G$, $T_n \subseteq G$ and that $f_{n,m}$ is defined by (2.6). Then, for all $m \geq m_0$,*

$$\|f - f_{n,m}\| \leq \|f - \mathcal{Q}_n f\| + \frac{k_n D_{n,m}}{C_{n,m}} \|f - \mathcal{Q}_n f\|, \quad (2.17)$$

where $k_n = \sup_{\substack{\phi \in T_n \\ \|\phi\|=1}} \|\phi\|$ and $C_{n,m}$, $D_{n,m}$ are given by (2.11) and (2.13) respectively.

Proof. Let $e_{n,m} = f_{n,m} - \mathcal{Q}_n f$ once more. Since $e_{n,m} \in T_n$, it follows from the definition of k_n and the inequality (2.16) that

$$\|e_{n,m}\| \leq k_n \|e_{n,m}\| \leq \frac{k_n D_{n,m}}{C_{n,m}} \|f - \mathcal{Q}_n f\|.$$

The full result is obtained from the triangle inequality $\|f - f_{n,m}\| \leq \|e_{n,m}\| + \|f - \mathcal{Q}_n f\|$. \square

Remark 1 In practice, it is useful to have an upper bound for the constant k_n . A simple exercise gives $k_n \leq \sum_{j=1}^n \|\Phi_j\|$, where $\{\Phi_j\}_{j=1}^n$ is any orthonormal basis for T_n .

Theorem 2.4 confirms that the approximation $f_{n,m}$ is quasi-optimal whenever m is sufficiently large in comparison to n . In other words, the error $\|f - f_{n,m}\|$ can be bounded by a constant multiple (independent of n) of $\|f - \mathcal{Q}_n f\|$. Naturally, whenever $\{\phi_1, \dots, \phi_n\}$ are the first n vectors in an infinite sequence ϕ_1, ϕ_2, \dots that form a basis for H (a natural assumption to make from a practical standpoint), we find that $f_{n,m}$ converges to f at the same rate as $\mathcal{Q}_n f$, provided m scales appropriately with n . Under the same assumptions, Corollary 2.6 also verifies convergence of $f_{n,m}$ to f in $\|\cdot\|$, whenever $\mathcal{Q}_n f \rightarrow f$ in this norm and $k_n \|f - \mathcal{Q}_n f\| \rightarrow 0$ as $n \rightarrow \infty$.

Note on other feature of this framework. Provided m is such that $C_{n,m} > 0$, any function $f \in T_n$ will be recovered exactly by $f_{n,m}$. In the language of sampling theory, this property is commonly referred to as *perfect* reconstruction.

Remark 2 Whilst this framework relies on the fact that m can range independently of n , a natural question to ask is what happens if m is set equal to n (note that, in this case, one recovers the well-known *consistent sampling* framework of Eldar et al [21, 47]). This question was discussed in detail in [3], where it demonstrated that such an approach often leads to severe ill-conditioning as $n = m \rightarrow \infty$. Additionally, stringent restrictions are placed on the types of vectors f that can be reconstructed (see also Section 3.6). Conversely, by allowing m to vary independently of n , we obtain a reconstruction $f_{n,m}$ that is guaranteed to converge for any vector $f \in H$. Moreover, as we discuss in Section 2.3, provided the reconstruction vectors are suitably chosen, the computation of $f_{n,m}$ is completely stable.

2.1 The case of an orthogonal sampling basis

In the previous setup, the sampling basis was assumed to be a Riesz basis. The particular case of $\{\psi_j\}_{j=1}^\infty$ being an orthonormal basis warrants further study, since this situation often arises in applications (e.g. Fourier sampling).

In this setting, both (2.1) and (2.2) (which are now identical) hold with equality and with all constants being precisely one. In other words, Parseval's identity

$$\|f\|^2 = \sum_{j=1}^{\infty} |\langle f, \psi_j \rangle|^2, \quad \forall f \in H,$$

holds. Our first result provides a geometric interpretation for the constant $C_{n,m}$ in terms of subspace angles. Recall that if U and V are subspaces of H , then the angle θ_{UV} is given by

$$\cos(\theta_{UV}) = \inf_{\substack{u \in U \\ \|u\|=1}} \|\mathcal{Q}_V u\|, \quad (2.18)$$

where $\mathcal{Q}_V : H \rightarrow V$ is the orthogonal projection. We have

Lemma 2.7. *Suppose that $\{\psi_j\}_{j=1}^\infty$ is an orthonormal basis of H . Then $C_{n,m} = \cos^2(\theta)$, where $\theta = \theta_{T_n S_m}$ is the angle between the subspaces T_n and S_m .*

Proof. Whenever $\{\psi_j\}_{j=1}^\infty$ is orthonormal, the operator \mathcal{P}_m is the orthogonal projection onto S_m . Hence the result follows immediately from (2.11) and (2.18). \square

Whilst this property is interesting, the following result has much greater significance:

Theorem 2.8. *Suppose that $\{\psi_j\}_{j=1}^\infty$ is an orthonormal basis of H . Then, for all $m \geq m_0$, the approximation $f_{n,m}$ satisfies*

$$\|f - \mathcal{Q}_n f\| \leq \|f - f_{n,m}\| \leq K_{n,m} \|f - \mathcal{Q}_n f\| \quad (2.19)$$

where

$$K_{n,m} = \sqrt{1 + (1 - C_{n,m})C_{n,m}^{-2}} = \sqrt{1 + \tan^2 \theta \sec^2 \theta},$$

and θ is as in Lemma 2.7. In particular, for fixed n , $K_{n,m} \rightarrow 1$ as $m \rightarrow \infty$, and hence $f_{n,m} \rightarrow \mathcal{Q}_n f$ as $m \rightarrow \infty$.

Proof. Note that $\mathcal{P} = \mathcal{I}$ for an orthonormal sampling basis. In particular, $\epsilon_{n,m} = C_n^* - C_{n,m}$ and $C_n^* = 1$. Hence, using Lemma 2.3 we find that $D_{n,m}^2 \leq 1 - C_{n,m}$. The result now follows from Theorem 2.4. \square

From this we conclude the following: not only is $f_{n,m}$ quasi-optimal, it is also *asymptotically optimal* in the sense that $f_{n,m} \rightarrow \mathcal{Q}_n f$, the best approximation to f from T_n , as $m \rightarrow \infty$. Hence, using this approach, we can recover an approximation to f that is arbitrarily close to the error minimising approximant (which, as mentioned, cannot be computed directly from the given samples). Moreover, the rate of convergence of $f_{n,m}$ to $\mathcal{Q}_n f$ is completely independent of the particular vector f , and relies only on rate of decay of the parameter $1 - C_{n,m}$.

Note that asymptotic optimality also occurs for general Riesz bases whenever $\mathcal{P}(T_n) \subseteq T_n$, in which case $D_{n,m} \rightarrow 0$ as $m \rightarrow \infty$ (see Lemma 2.3). The case of orthonormal sampling vectors presents the most obvious example of a basis satisfying this condition.

Remark 3 Whenever the vectors $\{\psi_j\}_{j=1}^\infty$ are not orthonormal, a natural question to ask is whether we can modify the approach for computing $f_{n,m}$ to recover asymptotic optimality. This can be easily done, at least in theory, by replacing the operator \mathcal{P}_m , given by (2.4), with the orthogonal projection $H \rightarrow T_n$. In this case, both Lemma 2.7 and Theorem 2.8 will hold for nonorthogonal sampling bases. The downside of the approach is that it requires additional computational cost to compute $f_{n,m}$, as we explain at the end of the next section.

Another potential means to recover asymptotic optimality is to define $\mathcal{P}_m g = \sum_{j=1}^m \langle g, \psi_j \rangle \psi_j^*$, where $\{\psi_j^*\}$ is the set of dual vectors to the sampling vectors $\{\psi_j\}$. In this case, $\mathcal{P}_m \rightarrow \mathcal{I}$ strongly, and asymptotic optimality follows. In practice, however, one may not have access to the dual vectors, thus this approach cannot necessarily be readily implemented.

2.2 Oblique asymptotic optimality

Whenever the sampling basis is orthonormal, $f_{n,m}$ converges to the best approximation $\mathcal{Q}_n f$ as $m \rightarrow \infty$. An obvious question to ask is what can be said in the general case, where the vectors $\{\psi_j\}$ only form a Riesz basis? The intuitive explanation given previously indicated that $f_{n,m} \approx \tilde{f}_n$, where \tilde{f}_n is defined by (2.8). We now wish to confirm this observation. In fact, as in the orthonormal case, we may demonstrate a stronger result: namely, for fixed $n \in \mathbb{N}$, $f_{n,m} \rightarrow \tilde{f}_n$ as $m \rightarrow \infty$ at a rate that is independent of the particular vector $f \in H$.

Recall that the form $a(\cdot, \cdot)$ yields an equivalent inner product on H . Since \tilde{f}_n is defined by the equations $a(\tilde{f}_n, \phi) = a(f, \phi)$, $\forall \phi \in T_n$, the mapping $f \mapsto \tilde{f}_n$ is the orthogonal projection onto T_n with respect to this inner product. Letting $\|g\|_a = \sqrt{a(g, g)}$ be the corresponding norm on H , we now define the constants

$$\tilde{C}_{n,m} = \inf_{\substack{\phi \in T_n \\ \|\phi\|_a = 1}} \langle \mathcal{P}_m \phi, \phi \rangle, \quad \tilde{D}_{n,m} = \sup_{\substack{f \in T_n^\perp \\ \|f\|_a = 1}} \sup_{\substack{g \in T_n \\ \|g\|_a = 1}} |\langle \mathcal{P}_m f, g \rangle|. \quad (2.20)$$

In this instance, T_n^\perp is defined with respect to the a -inner product, i.e. $T_n^\perp = \{f \in H : a(f, \phi) = 0, \forall \phi \in T_n\}$. Conversely, when considered with respect to the canonical inner product, this subspace is precisely $\mathcal{P}(T_n)^\perp = \{f \in H : \langle f, \phi \rangle = 0, \forall \phi \in \mathcal{P}(T_n)\}$.

Note the similarity between $\tilde{C}_{n,m}$ and $\tilde{D}_{n,m}$ and the quantities $C_{n,m}$ and $D_{n,m}$ defined in (2.11) and (2.13) respectively. Roughly speaking, the former measure the deviation of $f_{n,m}$ from $\mathcal{Q}_n f$, whereas, as we will subsequently show, the latter determine the deviation of $f_{n,m}$ from \tilde{f}_n .

With these definitions to hand, identical arguments to those given in the proofs of Lemmas 2.2 and 2.3 now yield:

Lemma 2.9. *For all $m, n \in \mathbb{N}$, $\tilde{C}_{n,m} \geq \frac{1}{d_2} C_{n,m}$, where $C_{n,m}$ is as in (2.11). Moreover, for fixed n , $\tilde{C}_{n,m} \rightarrow 1$ as $m \rightarrow \infty$.*

Lemma 2.10. *For all $m, n \in \mathbb{N}$, $\tilde{D}_{n,m} \leq d_2$ and $\tilde{D}_{n,m}^2 \leq c_2(1 - \tilde{C}_{n,m})$. In particular, for fixed n , $\tilde{D}_{n,m} \rightarrow 0$ as $m \rightarrow \infty$.*

Using these lemmas, we deduce

Corollary 2.11. *If $f_{n,m}$ and \tilde{f}_n are given by (2.6) and (2.8) respectively, then*

$$\|f_{n,m} - \tilde{f}_n\|_a \leq \frac{\tilde{D}_{n,m}}{\tilde{C}_{n,m}} \|f - \tilde{f}_n\|_a,$$

and we have the error estimate

$$\|f - \tilde{f}_n\|_a \leq \|f - f_{n,m}\|_a \leq \tilde{K}_{n,m} \|f - \tilde{f}_n\|_a, \quad \tilde{K}_{n,m} = \sqrt{1 + \tilde{D}_{n,m}^2 \tilde{C}_{n,m}^{-2}}.$$

In particular, for any $f \in \mathcal{H}$, $f_{n,m} \rightarrow \tilde{f}_n$ as $m \rightarrow \infty$.

Proof. Since $(f_{n,m} - \tilde{f}_n) \in \mathcal{T}_n$, we have

$$\tilde{C}_{n,m} \|f_{n,m} - \tilde{f}_n\|_a^2 \leq \langle \mathcal{P}_m(f_{n,m} - \tilde{f}_n), f_{n,m} - \tilde{f}_n \rangle.$$

Moreover, because $\langle \mathcal{P}_m f_{n,m}, \phi \rangle = \langle \mathcal{P}_m f, \phi \rangle$, we deduce that

$$\tilde{C}_{n,m} \|f_{n,m} - \tilde{f}_n\|_a^2 \leq \langle \mathcal{P}_m(f - \tilde{f}_n), f_{n,m} - \tilde{f}_n \rangle \leq \tilde{D}_{n,m} \|f - \tilde{f}_n\|_a \|f_{n,m} - \tilde{f}_n\|_a,$$

where the second inequality follows from the definition (2.20) of $\tilde{D}_{n,m}$ and the fact that $(f - \tilde{f}_n) \in \mathcal{T}_n^\perp$, the orthogonal complement of \mathcal{T}_n with respect to the a -inner product. \square

Note that the mapping $\mathcal{W}_n : f \mapsto \tilde{f}_n$ is an *oblique* projection with respect to the inner product $\langle \cdot, \cdot \rangle$ on \mathcal{H} . In particular, \mathcal{W}_n has range \mathcal{T}_n and kernel $\mathcal{P}(\mathcal{T}_n)^\perp$, and we have the decomposition $\mathcal{H} = \mathcal{T}_n \oplus \mathcal{P}(\mathcal{T}_n)^\perp$. For this reason, we say that $f_{n,m}$ possesses *oblique asymptotic* optimality.

2.3 Computation of $f_{n,m}$

Recall that the computation of the approximation $f_{n,m}$ involves solving the system of equations (2.6). This can be interpreted as the normal equations of a least squares problem. Suppose that $f_{n,m} = \sum_{j=1}^n \alpha_j \phi_j$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $\hat{f} = (\hat{f}_1, \dots, \hat{f}_m)$. If U is the $m \times n$ matrix with $(j, k)^{\text{th}}$ entry $\langle \phi_k, \psi_j \rangle$, then (2.6) is given exactly by $A\alpha = U^\dagger \hat{f}$, where $A = U^\dagger U$ and U^\dagger is the adjoint of U . In other words, the vector α is the least squares solution of the problem $U\alpha \approx \hat{f}$.

This system can be solved iteratively by applying conjugate gradient iterations to the normal equations, for example. The number of required iterations is dependent on the condition number $\kappa(A)$ of the matrix A . Specifically, the number of iterations required to obtain numerical convergence (i.e. to within a prescribed tolerance) is proportional to $\sqrt{\kappa(A)}$ [26]. In particular, if $\kappa(A)$ is $\mathcal{O}(1)$ for all n and $m \geq m_0$, then the number of iterations is also $\mathcal{O}(1)$ for all n . Hence, the cost of computing $f_{n,m}$ is determined solely by the number of operations required to perform matrix-vector multiplications involving U . In other words, only $\mathcal{O}(mn)$ operations.

Naturally, aside from this consideration, the condition number of A is also important since it determines susceptibility of the numerical computation to both round-off error and noise. Specifically, an error of magnitude ϵ in the inputs (i.e. the samples \hat{f}_j , $j = 1, \dots, m$) will yield an error of magnitude roughly $\kappa(A)\epsilon$ in the output $f_{n,m}$.

For these reasons it is of utmost importance to study the condition number of A . For this, we first introduce the Hermitian matrix $\tilde{A} \in \mathbb{C}^{n \times n}$ with $(j, k)^{\text{th}}$ entry $\langle \phi_j, \phi_k \rangle$. Note that \tilde{A} is the Gram matrix of the vectors $\{\phi_1, \dots, \phi_n\}$. In particular, $\kappa(\tilde{A})$ is a measure of the suitability of the particular vectors in which to compute $\mathcal{Q}_n f$. With \tilde{A} to hand, we also introduce the related matrix $\tilde{A}_a \in \mathbb{C}^{n \times n}$ with $(j, k)^{\text{th}}$ entry $a(\phi_j, \phi_k) = \langle \mathcal{P}\phi_j, \phi_k \rangle$, i.e. the Gram matrix with respect to the inner product $a(\cdot, \cdot)$.

The following lemma comes as no surprise:

Lemma 2.12. *The matrices \tilde{A} and \tilde{A}_a are spectrally equivalent. In particular, for all $n \in \mathbb{N}$,*

$$\frac{d_1}{d_2} \kappa(\tilde{A}) \leq \kappa(\tilde{A}_a) \leq \frac{d_2}{d_1} \kappa(\tilde{A}).$$

Proof. For any Hermitian matrix B , the condition number is the ratio of the largest and smallest eigenvalues (in absolute value). Moreover, if B is positive definite, then

$$\inf_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \left\{ \frac{\alpha^\dagger B \alpha}{\alpha^\dagger \alpha} \right\} = \lambda_{\min}(B), \quad \sup_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \left\{ \frac{\alpha^\dagger B \alpha}{\alpha^\dagger \alpha} \right\} = \lambda_{\max}(B). \quad (2.21)$$

If $\phi = \sum_{j=1}^n \alpha_j \phi_j$, then $\alpha^\dagger \tilde{A} \alpha = \|\phi\|^2$ and $\alpha^\dagger \tilde{A}_a \alpha = a(\phi, \phi)$. Hence, spectral equivalence now follows immediately from (2.10). \square

Concerning the condition number of the matrix A , we now have the following:

Lemma 2.13. *Suppose that $m \geq m_0$, where m_0 is as in Theorem 2.4, and $\tilde{C}_{n,m}$ and $C_{n,m}$ are given by (2.11) and (2.20) respectively. Then*

$$\tilde{C}_{n,m} \kappa(\tilde{A}_a) \leq \kappa(A) \leq \frac{1}{\tilde{C}_{n,m}} \kappa(\tilde{A}_a), \quad \frac{C_{n,m}}{d_2} \kappa(\tilde{A}) \leq \kappa(A) \leq \frac{d_2}{C_{n,m}} \kappa(\tilde{A}).$$

Moreover, for fixed n , $A \rightarrow \tilde{A}_a$ as $m \rightarrow \infty$, and, if $\mathcal{P} = \mathcal{I}$, $A \rightarrow \tilde{A} = \tilde{A}_a$.

Proof. The matrix A is Hermitian and, provided $m \geq m_0$, positive definite. Hence, its eigenvalues are given by (2.21). For $\phi = \sum_{j=1}^n \alpha_j \phi_j$, we have $\alpha^\dagger A \alpha = \langle \mathcal{P}_m \phi, \phi \rangle$. By definition of $C_{n,m}$ and $\tilde{C}_{n,m}$, we find that $\lambda_{\min}(A) \geq C_{n,m} \lambda_{\min}(\tilde{A})$ and $\lambda_{\min}(A) \geq \tilde{C}_{n,m} \lambda_{\min}(\tilde{A}_a)$. Moreover, by (2.1) we have $\lambda_{\max}(A) \leq d_2 \lambda_{\max}(\tilde{A})$ and $\lambda_{\max}(A) \leq \lambda_{\max}(\tilde{A}_a)$. The first result now follows immediately from (2.21). For the second, we merely note that each entry of A converges to the corresponding entry of \tilde{A}_a as $m \rightarrow \infty$. \square

Note the important conclusion of this lemma: computing $f_{n,m}$ from (2.6) is no more ill-conditioned than the computation of the orthogonal projection $\mathcal{Q}_n f$ or the oblique projection $\mathcal{W}_n f$ in terms of the vectors $\{\phi_1, \dots, \phi_n\}$. In practice, it is often true that these vectors correspond to the first n vectors in a basis $\{\phi_j\}_{j=1}^\infty$ of \mathcal{H} with additional structure. Whenever this is the case, as the following trivial corollary indicates, we can expect good conditioning:

Corollary 2.14. *Suppose that $\{\phi_j\}_{j=1}^\infty$ is a Riesz basis for \mathcal{H} (with respect to $\langle \cdot, \cdot \rangle$) with constants c'_1 and c'_2 . Then*

$$\kappa(A) \leq \frac{c'_2 d_2}{c'_1 C_{n,m}}.$$

Proof. This follows immediately from (2.1) and Lemma 2.13. \square

Put together, a fundamental conclusion of Theorem 2.4, Lemma 2.13 and Corollary 2.14 is the following: for a given reconstruction space T_n , the individual vectors ϕ_1, \dots, ϕ_n can be chosen arbitrarily, without altering the analysis of the approximation $f_{n,m}$ (which itself does not depend on the individual vectors used to represent it). The choice of vectors only becomes important when considering the condition number of linear system to solve. Moreover, the quality of a system of vectors for the reconstruction problem is completely intrinsic, in that it is determined only by the corresponding Gram matrix (in particular, it is independent of the sampling space).

Corollary 2.14 confirms that the approximation $f_{n,m}$ can be readily computed in a stable manner for many choices of reconstruction basis. However, to fully implement this method, as we discuss further in the next section, it is useful to have numerical way of computing $C_{n,m}$. The following lemma provides such a means:

Lemma 2.15. *The quantity $C_{n,m}$ is given by $C_{n,m} = \lambda_{\min}(\tilde{A}^{-1}A)$. Moreover, if \tilde{A} and A commute, then $C_{n,m} = 1 - \|I - \tilde{A}^{-1}A\|$. In particular, if $\{\phi_j\}_{j=1}^n$ is an orthonormal basis, then $C_{n,m} = \lambda_{\min}(A) = 1 - \|I - A\|$.*

Proof. By definition $C_{n,m} = \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} \langle \mathcal{P}_m \phi, \phi \rangle$. Letting $\phi = \sum_{j=1}^n \alpha_j \phi_j$, we find that

$$C_{n,m} = \inf_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \frac{\sum_{j,k=1}^n \alpha_j \bar{\alpha}_k \langle \mathcal{P}_m \phi_j, \phi_k \rangle}{\sum_{j,k=1}^n \alpha_j \bar{\alpha}_k \langle \phi_j, \phi_k \rangle} = \inf_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \frac{\alpha^\dagger A \alpha}{\alpha^\dagger \tilde{A} \alpha}.$$

We now claim that, for arbitrary Hermitian positive definite matrices B and C with B nonsingular, the following holds:

$$\inf_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \frac{\alpha^\dagger C \alpha}{\alpha^\dagger B \alpha} = \lambda_{\min}(B^{-1}C), \quad \sup_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \frac{\alpha^\dagger C \alpha}{\alpha^\dagger B \alpha} = \lambda_{\max}(B^{-1}C).$$

To do so, write $B = D^\dagger D$, with D nonsingular. Then, after rearranging, we obtain

$$\inf_{\substack{\alpha \in \mathbb{C}^n \\ \alpha \neq 0}} \frac{\alpha^\dagger C \alpha}{\alpha^\dagger B \alpha} = \inf_{\substack{\beta \in \mathbb{C}^n \\ \beta \neq 0}} \frac{\beta^\dagger D^{-\dagger} C D^{-1} \beta}{\beta^\dagger \beta} = \lambda_{\min}(D^{-\dagger} C D^{-1}),$$

for example. However, a trivial calculation confirms that the eigenvalues of $D^{-\dagger} C D^{-1}$ are identical to those of $B^{-1}C$, thus establishing the claim. Since \tilde{A} is nonsingular, this confirms that $C_{n,m} = \lambda_{\min}(\tilde{A}^{-1}A)$. For the second result, we merely notice that $\lambda_{\min}(B) = 1 - \lambda_{\max}(I - B) = 1 - \|I - B\|$, whenever B is Hermitian. \square

In Section 2.1, we briefly discussed a modified approach where the operator \mathcal{P}_m , usually given by (2.4), was replaced by the orthogonal projection operator. The advantage of this approach is that it guarantees asymptotic optimality. However, the downside is additional computational expense. Indeed, the corresponding matrix is of the form $A = U^\dagger V^{-1}U$, where $V \in \mathbb{C}^{m \times m}$ has $(j, k)^{\text{th}}$ entry $\langle \psi_j, \psi_k \rangle$. Hence, if conjugate gradients iterations are used, at each stage we are required to compute matrix-vector products involving the $m \times m$ matrix V^{-1} (assuming that V^{-1} had been precomputed). In general, this requires $\mathcal{O}(m^2)$ operations. Thus, we incur a cost of $\mathcal{O}(m^2)$, as opposed to $\mathcal{O}(mn)$ for the original algorithm. Hence, in practice it may be better settle for only quasi- and oblique asymptotic optimality, whilst retaining a lower computational cost.

2.4 Conditions for guaranteed, quasi-optimal recovery

Let us return to the standard form of the method once more. To implement this method, it is necessary to have conditions that guarantee nonsingularity, stability and quasi-optimal recovery. In other words, for given sampling and reconstruction bases, we wish to study the quantity

$$\Theta(n; \theta) = \min \{m \in \mathbb{N} : C_{n,m} \geq \theta\}, \quad \theta \in (0, d_2), \quad (2.22)$$

where $C_{n,m}$ is given by (2.11) and d_2 stems from (2.2). Note that $C_{n,m} \leq d_2$ by (2.2), thereby explaining the stated range of θ . Also, by Lemma 2.2, we have that $\lim_{m \rightarrow \infty} C_{n,m} \geq d_1 > 0$, thus Θ is well-defined.

By definition, $\Theta(n; \theta)$ is the least m such that $\|f - f_{n,m}\| \leq c(\theta)\|f - \mathcal{Q}_n f\|$, where

$$c(\theta) = \sqrt{1 + d_2^2 \theta^{-2}} \quad \text{or} \quad \sqrt{1 + (1 - \theta)\theta^{-2}}, \quad (2.23)$$

whenever the sampling basis is orthonormal. In other words, the least m required for quasi-optimal recovery with constant $c(\theta)$. Thus, provided $m \geq \Theta(n; \theta)$, the approximation $f_{n,m}$ converges at the same rate as $\mathcal{Q}_n f$ as $n \rightarrow \infty$. In addition, $m \geq \Theta(n; \theta)$ guarantees that $\|f_{n,m}\| \leq d_2 \theta^{-1} \|f\|$ and $\kappa(A) \leq d_2 \theta^{-1} \kappa(\tilde{A})$, thus making the linear system for $f_{n,m}$ solvable in a number of operations proportional to $\sqrt{d_2 \theta^{-1} \kappa(\tilde{A})}$.

Note that $\Theta(n; \theta)$ is determined only by the sampling and reconstruction spaces S_m and T_n . Whilst $\Theta(n; \theta)$ can be numerically computed for any pair of spaces via the expression given in Lemma 2.15, analytical bounds must be determined on a case-by-case basis. In the next section, where we consider the recovery of functions from their Fourier samples using (piecewise) polynomial bases, we are able to derive explicit forms for such bounds.

Remark 4 As mentioned, the framework developed in this section was first introduced by the authors in [3]. Whilst a result similar to Theorem 2.4 was proved, there are a number of important improvements offered by the theory presented in this paper:

1. In [3] it was assumed that the reconstruction vectors ϕ_1, \dots, ϕ_n were the first n in an infinite sequence of vectors that formed a Riesz basis for H . Conversely, Theorem 2.4 depends only on the subspace T_n , and thus the individual reconstruction vectors can be chosen arbitrarily.
2. The constants $K_{n,m}$ and $C_{n,m}$ are known exactly in terms of the sampling and reconstruction bases, and, whenever the sampling vectors are orthogonal, they possess a simple geometric interpretation in terms of subspace angles. Moreover, these constants can be computed by determining either the minimal eigenvalue or, in certain cases, the norm of an $n \times n$ matrix.
3. Simple, explicit bounds for the condition number of the matrix A are known in terms of the constant $C_{n,m}$ and the Gram matrix \tilde{A} .
4. The behaviour of $f_{n,m}$ as $m \rightarrow \infty$ (for n fixed) can be fully explained in terms of oblique asymptotic optimality.

3 Polynomial reconstructions from Fourier samples

One of the most important examples of this procedure is the reconstruction of an analytic, but nonperiodic function f to high accuracy from its Fourier coefficients. Direct expansion in Fourier series converges only slowly in the $L^2(-1, 1)$ norm, and suffers from the Gibbs phenomenon near the domain boundary. Hence, given the first m Fourier coefficients of f , we now seek to reconstruct f to high accuracy in another basis.

Let $H = L^2(-1, 1)$ (the space of square integrable functions on $(-1, 1)$), $f : (-1, 1) \rightarrow \mathbb{R}$ and

$$\psi_j(x) = \frac{1}{\sqrt{2}} e^{ij\pi x}, \quad j \in \mathbb{Z},$$

be the standard Fourier basis functions. For $m \geq 2$, we assume that the coefficients

$$\hat{f}_j = \int_{-1}^1 f(x) \overline{\psi_j(x)} dx, \quad j = -\left\lfloor \frac{m}{2} \right\rfloor + 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor - 1,$$

are known (note that, whenever m is even, this means that the first $m-1$ Fourier coefficients of f are given. We will allow this minor discrepancy since it simplifies ensuing analysis). As a consequence of Theorem 2.4, we are free to choose the reconstruction space. The orthogonal projection of an analytic function onto the space \mathbb{P}_{n-1} of polynomials of degree less than n is known to converge exponentially fast at rate ρ^{-n} , where $\rho > 1$ is determined from the largest Bernstein ellipse within which f is analytic [9]. Hence, we let $T_n = \mathbb{P}_{n-1}$. Note that an orthonormal basis for T_n is given by the functions

$$\phi_j(x) = \sqrt{j + \frac{1}{2}} P_j(x), \quad j \in \mathbb{N}, \quad (3.1)$$

where P_j is the j^{th} Legendre polynomial. Moreover, if \mathcal{Q}_n is the orthogonal projection onto T_n , then it is well-known that

$$\|f - \mathcal{Q}_n f\| \leq c_f \sqrt{n} \rho^{-n}, \quad (3.2)$$

where c_f depends only on the maximal value of f on the Bernstein ellipse indexed by ρ . Naturally, we could also assume finite regularity of f throughout, with suitable adjustments made to the various error estimates. However, for simplicity we shall not do this.

With this to hand, provided $m \geq \Theta(n; \theta)$ for some $\theta > 0$, where $\Theta(n; \theta)$ is defined in (2.22), the approximation $f_{n,m}$ obtained from the reconstruction procedure satisfies $\|f - f_{n,m}\| \leq c(\theta) \|f - \mathcal{Q}_n f\|$ (see Theorem 2.4). In particular, $\|f - f_{n,m}\| \leq c(\theta) c_f \sqrt{n} \rho^{-n}$. Hence, exponential convergence of $f_{n,m}$. The key question remaining is how large m must be in comparison to n to ensure such rates of convergence. Resolving this question involves estimating the quantity $C_{n,m}$, a task we next pursue.

3.1 Estimates for $\Theta(n; \theta)$

For both numerical and analytical estimates of $\Theta(n; \theta)$ we need to select an appropriate basis of \mathbb{P}_{n-1} (recall from Section 2 that $\Theta(n; \theta)$ is independent of the basis used). A natural choice is the orthonormal basis (3.1) of scaled Legendre polynomials. Fortunately, in this case, the inner products $\langle \phi_k, \psi_j \rangle$ (i.e. the entries of the matrix U) are known in closed form:

$$\langle \phi_k, \psi_j \rangle = (-i)^k \sqrt{\frac{k + \frac{1}{2}}{j}} J_{k+\frac{1}{2}}(j\pi), \quad j \in \mathbb{Z}, \quad k \in \mathbb{N}, \quad (3.3)$$

where J_m is the Bessel function of first kind. This follows directly from

$$j_m(z) = \frac{1}{2} (-i)^m \int_{-1}^1 e^{izx} P_m(x) dx, \quad \forall z \in \mathbb{C}, \quad (3.4)$$

(see [1, 10.1.14]), where j_m is the spherical Bessel function of the first kind, given by

$$j_m(z) = \sqrt{\frac{\pi}{2z}} J_{m+\frac{1}{2}}(z).$$

With this to hand, we may compute $C_{n,m}$ (and, in turn, $\Theta(n; \theta)$) via the expression given in Lemma 2.15. In Figure 1 we display the functions $\Theta(n; \frac{1}{2})$ and $\Theta(n; \frac{1}{4})$ against n . Immediately, we witness quadratic growth of $\Theta(n; \theta)$ with n , a result we verify in this section. In doing so, we shall also derive an upper bound for $\Theta(n; \theta)$ in terms of n and θ . This gives an explicit, analytic condition for quasi-optimal recovery. Whilst such a bound is completely robust (in that it holds for all n), we notice from Figure 1 that $\Theta(n; \theta)$, when scaled by n^{-2} , quickly converges to an asymptotic limit. In

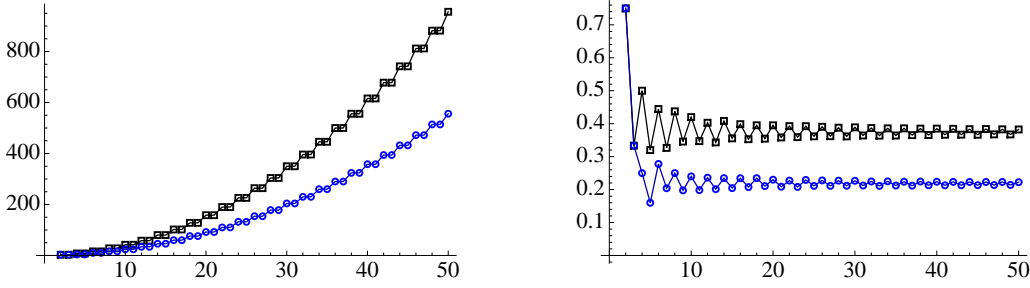


Figure 1: The functions $\Theta(n; \theta)$ (left) and $n^{-2}\Theta(n; \theta)$ (right) for $\theta = \frac{1}{2}$ (squares) and $\theta = \frac{1}{4}$ (circles).

practice it is wasteful to use a larger value of m than necessary (or, conversely, for fixed m a overly pessimistic value of n). Hence, in the second part of this section, we will also derive an asymptotic bound for $\Theta(n; \theta)$.

We commence as follows:

Lemma 3.1. *Suppose that $T_n = \mathbb{P}_{n-1}$, S_m is the space spanned by the first m Fourier basis functions and $m \geq \max\{2, \frac{2}{\pi}n\}$. Then $C_{n,m}$ satisfies*

$$C_{n,m} \geq 1 - \frac{4(\pi - 2)n^2}{\pi^2(2\lfloor \frac{m}{2} \rfloor - 1)}.$$

Proof. From the definition of $C_{n,m}$ and the fact that $\{\psi_j\}$ is an orthonormal basis we have

$$1 - C_{n,m} = 1 - \inf_{\substack{\phi \in T_n \\ \|\phi\|=1}} \langle \mathcal{P}_m \phi, \phi \rangle = \sup_{\substack{\phi \in T_n \\ \|\phi\|=1}} \langle \phi - \mathcal{P}_m \phi, \phi \rangle = \sup_{\substack{\phi \in T_n \\ \|\phi\|=1}} \|\phi - \mathcal{P}_m \phi\|^2,$$

where \mathcal{P}_m is the Fourier projection operator. It now follows that $1 - C_{n,m} \leq \sum_{k=0}^{n-1} \|\phi_k - \mathcal{P}_m \phi_k\|^2$, where ϕ_k is given by (3.1). By Parseval's theorem and the expression (3.3), we find that

$$\|\phi_k - \mathcal{P}_m \phi_k\|^2 = \sum_{|j| \geq \lfloor \frac{m}{2} \rfloor} \frac{k + \frac{1}{2}}{j} |J_{k+\frac{1}{2}}(j\pi)|^2.$$

Now, using a known result for Bessel functions [32], it can be shown that

$$\frac{k + \frac{1}{2}}{j} |J_{k+\frac{1}{2}}(j\pi)|^2 \leq \frac{2k+1}{j\pi \sqrt{j^2\pi^2 - (k + \frac{1}{2})^2}},$$

provided $j\pi > k + \frac{1}{2}$. Hence, for $m > \frac{2}{\pi}n$,

$$\|\phi_k - \mathcal{P}_m \phi_k\|^2 \leq \frac{2(2k+1)}{\pi^2} \sum_{j \geq \lfloor \frac{m}{2} \rfloor} \frac{1}{j \sqrt{j - \frac{(k+\frac{1}{2})^2}{\pi^2}}}. \quad (3.5)$$

Now, it was shown in [32] that $\sum_{j \geq m} \frac{1}{j \sqrt{j^2 - c^2}} \leq \frac{1}{c} \arcsin \frac{c}{m - \frac{1}{2}}$, whenever $m \geq c + \frac{1}{2}$. This gives

$$\|\phi_k - \mathcal{P}_m \phi_k\|^2 \leq \frac{4}{\pi} \arcsin \left[\frac{2k+1}{(2\lfloor \frac{m}{2} \rfloor - 1)\pi} \right],$$

and

$$1 - C_{n,m} \leq \frac{4}{\pi} \sum_{k=0}^{n-1} \arcsin \left[\frac{2k+1}{(2\lfloor \frac{m}{2} \rfloor - 1)\pi} \right].$$

We estimate this sum by the integral of $\arcsin t$. We have

$$1 - C_{n,m} \leq 2(2\lfloor \frac{m}{2} \rfloor - 1) \int_0^{\frac{2n}{(2\lfloor \frac{m}{2} \rfloor - 1)\pi}} \arcsin t \, dt.$$

Now, $\arcsin x \leq (\arcsin 1)x$ for $0 \leq x \leq 1$. It follows that $F(x) = \int_0^x \arcsin t \, dt \leq F(1)x^2$. Computing this integral exactly, we arrive at $F(1) = \frac{\pi}{2} - 1$. Upon substituting $x = \frac{2n}{(2\lfloor \frac{m}{2} \rfloor - 1)\pi}$, this completes the proof. \square

This lemma confirms that it is sufficient for m to scale quadratically with n for quasi-optimal recovery. Using this result, we find that

Theorem 3.2. *Suppose that T_n and S_m are as in Lemma 3.1. Then, for $n \geq 2$, $\Theta(n; \theta)$ satisfies*

$$\Theta(n; \theta) \leq 2 \left\lceil \frac{1}{2} + \frac{2(\pi - 2)}{\pi^2(1 - \theta)} n^2 \right\rceil, \quad \forall n \in \mathbb{N}.$$

Proof. Suppose that $m \geq \{2, \frac{2}{\pi}n\}$. Then, by Lemma 3.1, $C_{n,m} \geq \theta$ if

$$1 - \frac{4(\pi - 2)n^2}{\pi^2(2\lfloor \frac{m}{2} \rfloor - 1)} \geq \theta.$$

Rearranging, we find that

$$2 \left\lfloor \frac{m}{2} \right\rfloor \geq 1 + \frac{4(\pi - 2)n^2}{\pi^2(1 - \theta)} \Rightarrow m \geq 2 \left\lceil \frac{1}{2} + \frac{2(\pi - 2)}{\pi^2(1 - \theta)} n^2 \right\rceil$$

and the theorem is proved, provided the right-hand side exceeds $\max\{2, \frac{2}{\pi}\theta\}$. Since $n \geq 2$, the right-hand side is certainly greater than 2. Moreover,

$$1 + \frac{4(\pi - 2)n^2}{\pi^2(1 - \theta)} \geq \frac{8(\pi - 2)n}{\pi} > \frac{2n}{\pi},$$

as required. \square

Using a similar approach, we are also able to obtain an asymptotic bound for $\Theta(n; \theta)$ that is sharper than if we were to use Theorem 3.2 directly:

Theorem 3.3. *Suppose that T_n and S_m are as in Lemma 3.1. Then the function $\Theta(n; \theta)$ satisfies*

$$n^{-2}\Theta(n; \theta) \leq \frac{4}{\pi^2(1 - \theta)} + \mathcal{O}(n^{-2}), \quad n \rightarrow \infty.$$

Proof. Suppose that $m = cn^2$ and recall (3.5). Since $j < n$ and $k > \frac{1}{2}cn^2$, we deduce that

$$\|\phi_k - \mathcal{P}_m \phi_k\|^2 \leq \frac{2(2k+1)}{\pi^2} \sum_{j > \lfloor \frac{m}{2} \rfloor} \frac{1}{j^2} + \mathcal{O}(n^{-4}) = \frac{4(2k+1)}{c\pi^2 n^2} + \mathcal{O}(n^{-4}).$$

Hence

$$1 - C_{n, cn^2} \leq \frac{4}{c\pi^2 n^2} \sum_{k=0}^{n-1} (2k+1) + \mathcal{O}(n^{-2}) = \frac{4}{c\pi^2} + \mathcal{O}(n^{-2}).$$

Rearranging now gives the result. \square

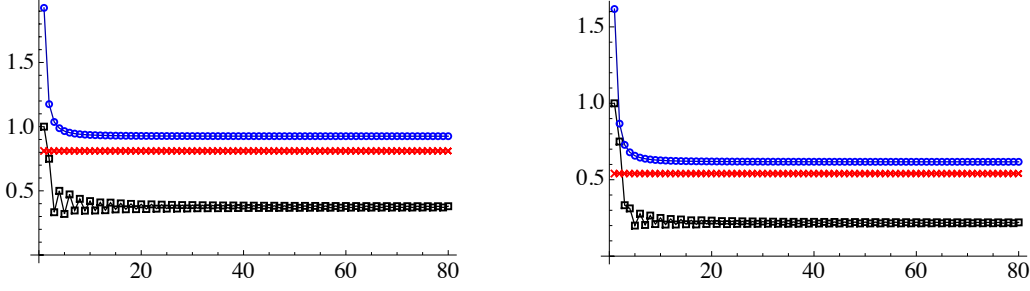


Figure 2: The function $n^{-2}\Theta(n; \theta)$ (squares), the global bound (circles) and the asymptotic bound (crosses), for $n = 2, \dots, 80$ and $\theta = \frac{1}{2}$ (left), $\theta = \frac{1}{4}$ (right).

In Figure 2 we compare the function $n^{-2}\Theta(n; \theta)$ for $\theta = \frac{1}{2}, \frac{1}{4}$ and the global and asymptotic bounds of Theorems 3.2 and 3.3. Note that both bounds are reasonably sharp in comparison to the computed values. In particular, as $n \rightarrow \infty$, $n^{-2}\Theta(n; \frac{1}{2})$ quickly approaches the limiting value $c \approx 0.38$, whereas the global and asymptotic upper bounds are 0.93 and 0.81 respectively.

At this moment, we reiterate an important point. Whilst Legendre polynomials were used in the proof of Lemma 3.1, the constant $C_{n,m}$ is independent of the particular reconstruction basis, and is only determined by the space T_n . Hence, Theorems 3.2 and 3.3 provide *a priori* estimates regardless of the particular implementation of the reconstruction procedure. In the next section, we discuss the choice of polynomial basis and its effect on the numerical method.

Remark 5 In some applications, medical imaging, for example, oversampling is common. Formally speaking, this is the situation where we wish to recover a function f with support in $[-1, 1]$ from its Fourier samples taken over an extended interval $K \supseteq [-1, 1]$ (e.g. $K = [-\frac{1}{\epsilon}, \frac{1}{\epsilon}]$ for some $0 < \epsilon \leq 1$). In this case, proceeding in a similar manner to before, we let $H = L^2(K)$,

$$\psi_j(x) = \sqrt{\frac{c}{2}} e^{icj\pi x}, \quad x \in K,$$

where $c = \frac{1}{2}|K|$ and $T_n = \{\phi : \phi|_{[-1,1]} \in \mathbb{P}_{n-1}, \text{supp}(\phi) \subseteq [-1, 1]\}$. Using similar arguments to those of Lemma 3.1, one can also derive estimates for $C_{n,m}$ and $\Theta(n; \theta)$ in this case. In fact,

$$C_{n,m} \geq 1 - \frac{4(\pi - 2)n^2}{c\pi^2(m - 1)}, \quad (3.6)$$

and

$$\Theta(n; \theta) \leq 2 \left[\frac{1}{2} + \frac{2(\pi - 2)}{c\pi^2(1 - \theta)} n^2 \right], \quad \forall n \in \mathbb{N}, \quad n^{-2}\Theta(n; \theta) \leq \frac{4}{c\pi^2(1 - \theta)} + \mathcal{O}(n^{-2}), \quad n \rightarrow \infty.$$

In particular, we retain the scaling $m = \mathcal{O}(n^2)$, regardless of the size of the interval K .

3.2 Choice of polynomial basis

The results proved in this section are independent of the polynomial basis used for implementation. In selecting such a basis, there are two questions which must be resolved. First, how stable is the resultant method, and second, how can the entries of the matrix U (as defined in Section 2.3) be computed? A straightforward choice is the orthogonal basis of Legendre polynomials (3.1). In this case, $\tilde{A} = I$, where \tilde{A} is the Gram matrix for $\{\phi_0, \dots, \phi_{n-1}\}$, making the method well-conditioned (Lemma 2.13). Moreover, the entries of U are known explicitly via (3.3).

Having said this, there is also interest in reconstructing in other polynomial bases. In many circumstances it may be advantageous to have an approximation $f_{n,m}$ that is easily manipulable. In this sense, an approximant composed of Legendre polynomials is not as convenient as one consisting of Chebyshev polynomials (of the first or second kind); the latter being easy to manipulate via the Fast Fourier Transform. To this end, the purpose of this section is to detail the implementation of this method in terms of general Gegenbauer polynomials.

Gegenbauer polynomials arise as orthogonal polynomials with respect to the inner product

$$\langle f, g \rangle_\lambda = \int_{-1}^1 f(x) \overline{g(x)} (1-x^2)^{\lambda-\frac{1}{2}} dx, \quad \lambda > -\frac{1}{2}.$$

For given λ , we denote the j^{th} such polynomial by $C_j^\lambda \in \mathbb{P}_j$. Important special cases are the Legendre polynomials ($\lambda = \frac{1}{2}$), and Chebyshev polynomials of the first ($\lambda = 0$) and second ($\lambda = 1$) kind. By convention [7] (see also [31]), each polynomial C_j^λ is normalised so that

$$C_j^\lambda(1) = \frac{\Gamma(j+2\lambda)}{j!\Gamma(2\lambda)}, \quad (3.7)$$

where Γ is the Gamma function, in which case it is known that (see [7, p.174])

$$\|C_j^\lambda\|_\lambda^2 = \sqrt{\pi} \frac{\Gamma(j+2\lambda)\Gamma(\lambda+\frac{1}{2})}{j!\Gamma(2\lambda)\Gamma(\lambda)(j+\lambda)}, \quad (3.8)$$

where $\|f\|_\lambda = \sqrt{\langle f, f \rangle_\lambda}$. With this to hand, we now define

$$\phi_j = \frac{1}{\|C_j^\lambda\|_\lambda} C_j^\lambda, \quad j = 0, 1, 2, \dots, \quad (3.9)$$

and seek to reconstruct f in this basis.

Our first task is to compute the entries of the matrix U . For this, we need to compute integrals of the form

$$I_k(z) = \int_{-1}^1 C_k^\lambda(x) e^{izx} dx, \quad k = 0, 1, 2, \dots,$$

where $z \in \mathbb{R}$. Fortunately, such integrals obey a simple recurrence relation:

Lemma 3.4. *For $z \neq 0$, the integrals $I_k(z)$ satisfy*

$$\begin{aligned} I_0(z) &= 2C_0^\lambda(1) \frac{\sin z}{z}, \quad I_1(z) = 2iC_1^\lambda(1) \frac{\sin z - z \cos z}{z^2}, \\ I_{k+1}(z) &= \frac{2i(k+\lambda)}{z} I_k(z) + I_{k-1}(z) - i \frac{e^{iz} + (-1)^k e^{-iz}}{z} [C_{k+1}^\lambda(1) - C_{k-1}^\lambda(1)], \quad k = 1, 2, \dots \end{aligned}$$

When $z = 0$, we have

$$I_0(0) = 2C_0^\lambda(1), \quad I_k(0) = \frac{1 + (-1)^k}{2(k+\lambda)} [C_{k+1}^\lambda(1) - C_{k-1}^\lambda(1)], \quad k = 1, 2, \dots$$

Proof. Recall the identity (see [7, p.176])

$$C_j^\lambda(x) = \frac{1}{2(j+\lambda)} \frac{d}{dx} [C_{j+1}^\lambda - C_{j-1}^\lambda], \quad j = 1, 2, \dots$$

Substituting this into the expression for $I_k(z)$ and integrating by parts gives

$$I_k(z) = \frac{1}{2(k+\lambda)} [(C_{k+1}^\lambda(x) - C_{k-1}^\lambda(x)) e^{izx}]_{x=-1}^1 - \frac{iz}{2(k+\lambda)} [I_{k+1}(z) - I_{k-1}(z)].$$

Rearranging now gives the general recurrence for $k \geq 1$. For $k = 0, 1$, we merely note that $C_0^\lambda(x) = C_0^\lambda(1)$, $C_1^\lambda(x) = C_1^\lambda(1)x$ and that

$$\int_{-1}^1 e^{izx} dx = 2 \frac{\sin z}{z}, \quad \int_{-1}^1 x e^{izx} dx = 2 \frac{\sin z - z \cos z}{z^2}.$$

The result for $z = 0$ is derived in a similar manner. \square

Using this recurrence formula, the matrix U can be formed in $\mathcal{O}(mn)$ operations. With this to hand, we now turn our attention to the condition number of \tilde{A} :

Theorem 3.5. *Let \tilde{A} be Gram matrix for the vectors $\{\phi_0, \dots, \phi_{n-1}\}$, where ϕ_j is given by (3.9). Then, $\kappa(\tilde{A}) = \mathcal{O}(n^{2\lambda-1})$ as $n \rightarrow \infty$. In particular, whenever $\phi_0, \dots, \phi_{n-1}$ arise from Chebyshev polynomials (of the first or second kinds), then $\kappa(\tilde{A}) = \mathcal{O}(n)$.*

To prove this theorem, we first require the following two lemmas. For convenience, we will write $L_\lambda^2(-1, 1)$, $\lambda > -\frac{1}{2}$, for the space of square-integrable functions with respect to the Gegenbauer weight function $(1-x^2)^{\lambda-\frac{1}{2}}$.

Lemma 3.6. *Suppose that $-\frac{1}{2} < \lambda < \frac{1}{2}$. Then, for all $g \in L^\infty(-1, 1)$, we have $\|g\| \leq \|g\|_\lambda$ and, for some $c_\lambda > 0$ independent of g ,*

$$\|g\|_\lambda \leq c_\lambda \|g\|^{\lambda+\frac{1}{2}} \|g\|_\infty^{\frac{1}{2}-\lambda}. \quad (3.10)$$

Conversely, if $\lambda \geq \frac{1}{2}$ then, for all $g \in L^\infty(-1, 1)$, $\|g\| \leq \|g\|_\lambda$ and

$$\|g\| \leq c_\lambda \|g\|_\lambda^{\frac{1}{\lambda+\frac{1}{2}}} \|g\|_\infty^{\frac{\lambda-\frac{1}{2}}{\lambda+\frac{1}{2}}}. \quad (3.11)$$

Proof. Suppose first that $-\frac{1}{2} < \lambda < \frac{1}{2}$. Trivially, $\|g\| \leq \|g\|_\lambda$. Now consider the other inequality. For any $0 < \epsilon < 1$, we have

$$\begin{aligned} \|g\|_\lambda^2 &= \int_{-1}^1 |g(x)|^2 (1-x^2)^{\lambda-\frac{1}{2}} dx \\ &= \int_{|x| \leq 1-\epsilon} |g(x)|^2 (1-x^2)^{\lambda-\frac{1}{2}} dx + \int_{1-\epsilon < |x| \leq 1} |g(x)|^2 (1-x^2)^{\lambda-\frac{1}{2}} dx \\ &\leq (1-(1-\epsilon)^2)^{\lambda-\frac{1}{2}} \|g\|^2 + 2\|g\|_\infty^2 \int_{1-\epsilon}^1 (1-x^2)^{\lambda-\frac{1}{2}} dx, \end{aligned}$$

where $\|\cdot\|_\infty$ is the uniform norm on $[-1, 1]$. Note that $(1-(1-y)^2)^{\lambda-\frac{1}{2}} < y^{\lambda-\frac{1}{2}}$, $\forall y \in (0, 1)$. It now follows that

$$\|g\|_\lambda^2 \leq \epsilon^{\lambda-\frac{1}{2}} \|g\|^2 + \frac{2}{\lambda+\frac{1}{2}} \epsilon^{\lambda+\frac{1}{2}} \|g\|_\infty^2, \quad 0 < \epsilon < 1.$$

Let $c > 2$ be arbitrary. Then $\|g\|^2 < c\|g\|_\infty^2$, so we may let $\epsilon = \frac{\|g\|_\infty^2}{c\|g\|_\lambda^2}$. Substituting this into the previous expression immediately gives (3.10).

Now suppose that $\lambda \geq \frac{1}{2}$. Once more, trivial arguments give that $\|g\|_\lambda \leq \|g\|$. For the other inequality, we proceed in a similar manner. We have $\|g\|^2 \leq \epsilon^{\frac{1}{2}-\lambda} \|g\|_\lambda^2 + 2\epsilon \|g\|_\infty^2$. For $c > 2$ we now set $\epsilon = \left(\frac{\|g\|_\lambda}{c\|g\|_\infty}\right)^{\frac{2}{\lambda+\frac{1}{2}}}$, which gives (3.11). \square

Lemma 3.7. *Let $\lambda \geq \frac{1}{2}$. Then, for all $\phi \in \mathbb{P}_{n-1}$, $\|\phi\|_\infty \leq k_{n,\lambda} \|\phi\|_\lambda$, where $k_{n,\lambda}$ depends only on n and λ and satisfies $k_{n,\lambda} = \mathcal{O}(n^{\lambda+\frac{1}{2}})$ as $n \rightarrow \infty$.*

Proof. Let $\{\phi_0, \dots, \phi_{n-1}\}$ be given by (3.9), and write an arbitrary $\phi \in \mathbb{P}_{n-1}$ as $\phi = \sum_{j=0}^{n-1} a_j \phi_j$, where $\sum_{j=0}^{n-1} |a_j|^2 = \|\phi\|_\lambda^2$. By the Cauchy-Schwarz inequality,

$$\|\phi\|_\infty^2 \leq \|\phi\|_\lambda^2 \sum_{j=0}^{n-1} \|\phi_j\|_\infty^2 = k_{n,\lambda}^2 \|\phi\|_\lambda^2.$$

We now wish to estimate $\|\phi_j\|_\infty$. Recall that $\|C_j^\lambda\|_\infty = C_j^\lambda(1)$ [7, p.206]. Hence, by (3.7) and (3.8), we have

$$\|\phi_j\|_\infty^2 = \frac{\Gamma(j+2\lambda)(j+\lambda)}{j! \sqrt{\pi} \Gamma(2\lambda) \Gamma(\lambda + \frac{1}{2})}.$$

Consider the ratio $\frac{\Gamma(j+2\lambda)}{j!}$. By Stirling's formula,

$$\frac{\Gamma(j+2\lambda)}{j!} = \mathcal{O}(j^{2\lambda-1}), \quad j \rightarrow \infty.$$

Hence $\|\phi_j\|_\infty^2 = \mathcal{O}(j^{2\lambda})$, which gives $k_{n,\lambda}^2 = \mathcal{O}(n^{2\lambda+1})$, as required. \square

Proof of Theorem 3.5. Since \tilde{A} is Hermitian and positive definite, its condition number is the ratio of its maximum and minimum eigenvalues. By a simple argument, we find that

$$\lambda_{\max}(\tilde{A}) = \sup_{\substack{\phi \in \mathbb{P}_{n-1} \\ \phi \neq 0}} \frac{\|\phi\|^2}{\|\phi\|_\lambda^2}, \quad \lambda_{\min}(\tilde{A}) = \inf_{\substack{\phi \in \mathbb{P}_{n-1} \\ \phi \neq 0}} \frac{\|\phi\|^2}{\|\phi\|_\lambda^2}.$$

Consider the case $\lambda > \frac{1}{2}$. By Lemma 3.6, we have $\lambda_{\min}(\tilde{A}) \geq 1$ and

$$\lambda_{\max}(\tilde{A}) \leq c_\lambda^2 \sup_{\substack{\phi \in \mathbb{P}_{n-1} \\ \phi \neq 0}} \left(\frac{\|\phi\|_\infty}{\|\phi\|_\lambda} \right)^{2 \frac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}}.$$

Using Lemma 3.7, we deduce that $\lambda_{\max}(\tilde{A}) = \mathcal{O}(n^{2\lambda-1})$, as required. For the case $-\frac{1}{2} < \lambda < \frac{1}{2}$, we proceed in a similar manner. \square

This theorem confirms that the method can be implemented using Chebyshev polynomials whilst incurring only a mild growth in the condition number. It follows that, if conjugate gradients are used to compute the approximation, the total computational cost of forming $f_{n,m}$ is $\mathcal{O}(mn^{\frac{3}{2}})$, as opposed to $\mathcal{O}(mn)$ in the Legendre polynomial case. In the next section we present several examples of this implementation.

Remark 6 Whilst Theorem 3.5 provides an asymptotic estimate for $\kappa(\tilde{A})$ (and hence $\kappa(A)$), it may also be useful to derive global bounds. With effort, one could obtain versions of Lemmas 3.6 and 3.7 involving explicit bounds. For the sake of brevity, we shall not do this. However, whenever Chebyshev polynomials are used (arguably the most important case), it is a relatively simple exercise to confirm that

$$\kappa(\tilde{A}) \leq 2\sqrt{2}n, \quad \kappa(\tilde{A}) \leq 3^{\frac{2}{3}} \pi^{-\frac{1}{3}} \left[n(n + \frac{1}{2})(n + 1) \right]^{\frac{1}{3}},$$

in the first and second kind cases respectively.

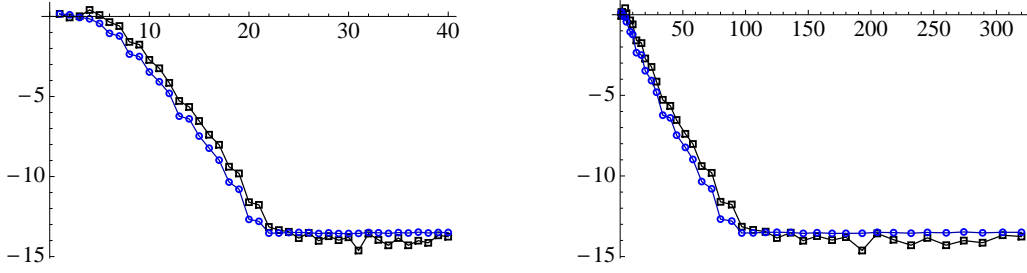


Figure 3: Error in approximating $f(x) = e^{-x} \cos 4x$ by $f_{n,m}(x)$ for $n = 1, \dots, 40$. Left: log error $\log_{10} \|f - f_{n,m}\|_{\infty}$ (squares) and $\log_{10} \|f - f_{n,m}\|$ (circles) against n . Right: log error against $m = 0.2n^2$.

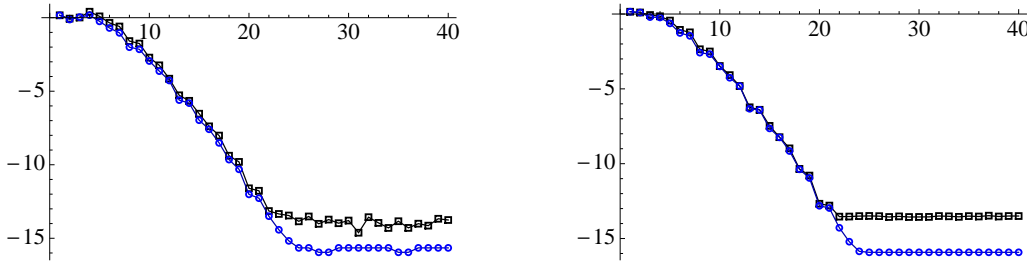


Figure 4: Error in approximating $f(x) = e^{-x} \cos 4x$ by $f_{n,m}(x)$ (squares) and $Q_n f(x)$ (circles) for $n = 1, \dots, 40$. Left: log uniform error. Right: log L^2 error.

3.3 Numerical examples

We now present several numerical examples of this method. All examples employ the value $m = 0.2n^2$, and the first series of examples consider the implementation using Legendre polynomials. In Figure 3 we consider the function $f(x) = e^{-x} \cos 4x$. Since f is analytic in this case, we witness exponential convergence in terms of n and root exponential convergence in terms of m . Note the effectiveness of the method: using less than 100 Fourier coefficients, we obtain an approximation with 13 digits of accuracy.

As indicated by Theorem 2.4, the approximation $f_{n,m}$ is quasi-optimal. To highlight this feature of the method, Figure 4 displays both the error in approximating f by $f_{n,m}$ and the best approximation $Q_n f$. Note the very close correspondence of the two graphs.

The example in Figures 3 and 4 is, in fact, entire. Hence, the approximation $f_{n,m}$ converges super-geometrically in n (as seen in Figure 3). For a meromorphic function, with complex singularity lying outside $[-1, 1]$, the convergence rate is truly exponential at a rate ρ . This is demonstrated in Figure 5, the approximated function being $f(x) = \frac{1}{1+x^2}$. Note that, despite the poles at $x = \pm i$, the approximation $f_{n,m}$ still obtains 13 digits of accuracy using only 250 Fourier coefficients.

Next we consider reconstructions in other polynomials using the work of the previous section. In Table 1 we give the error in approximating the function $f(x) = e^{-x} \cos 4x$ with Chebyshev polynomials of the first and second kinds. Note that the resulting uniform error is virtually identical to the case of the Legendre polynomial implementation (unsurprisingly, since all three implementations compute exactly the same approximation $f_{n,m}$, up to numerical error). Moreover, as evidenced by Table 2, the payoff is only mild growth in the condition number $\kappa(A)$.

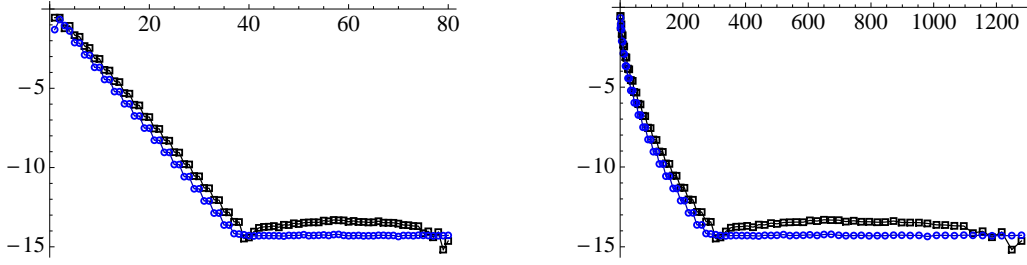


Figure 5: Error in approximating $f(x) = \frac{1}{1+x^2}$ by $f_{n,m}(x)$ for $n = 1, \dots, 80$. Left: log error $\log_{10} \|f - f_{n,m}\|_\infty$ (squares) and $\log_{10} \|f - f_{n,m}\|$ (circles) against n . Right: log error against $m = 0.2n^2$.

n	5	10	15	20	25	30	35	40
(a)	1.45e0	1.85e-3	3.03e-7	2.53e-12	1.06e-14	8.42e-14	4.06e-14	5.31e-14
(b)	1.45e0	1.85e-3	3.03e-7	2.53e-12	3.51e-14	1.16e-13	4.57e-14	7.70e-14
(c)	1.45e0	1.85e-3	3.03e-7	2.49e-12	6.76e-14	7.33e-14	6.40e-14	5.15e-14

Table 1: Comparison of the error $\|f - f_{n,m}\|_\infty$ with $m = 0.2n^2$, where $f_{n,m}$ is formed from (a) Legendre polynomials and Chebyshev polynomials of the (b) first and (c) second kinds.

3.4 Connections to earlier work

Rather than choosing m such that $C_{n,m} \geq \theta$, it may appear advantageous to find the minimum m such that $C_{n,m} > 0$. In other words, the smallest m such that $f_{n,m}$ is guaranteed to exist. Letting $\theta = 0$ in Theorems 3.2 and 3.3, we immediately obtain a sufficient condition of the form $m \geq cn^2$, for some $c > 0$. However, this result is far too pessimistic: it is known that reconstruction is always possible, provided $m \geq n$ [32]. For this reason, it may appear favourable to reconstruct using $m = n$, leading to the so-called *inverse polynomial reconstruction method* [37, 38]. Unfortunately, however, this approach is extremely unstable. The linear system has geometrically large condition number, making the procedure extremely sensitive to both noise and round-off error. Moreover, a continuous analogue of the Runge phenomenon occurs. Roughly speaking, the approximation $f_{n,m}$ only converges to f if geometric decay of $\|f - Q_n f\|$ is faster than the geometric growth of $\|A^{-1}\|$, meaning that only functions analytic in sufficiently large complex regions can be approximated by this procedure (as discussed in detail in [3], this behaviour can be understood in terms of the operator-theoretic properties of finite sections of certain non-Hermitian infinite matrices). On the other hand, by allowing m to range independently of n , we overcome all these difficulties, and obtain a stable method whose convergence is completely determined by the convergence of $Q_n f$ to f .

For the specific example of Legendre polynomial reconstructions from Fourier samples, this approach has also been recently considered in [32]. Therein, the estimate $m = \mathcal{O}(n^2)$ was derived, along with bounds for the error. Naturally, this problem is just one specific example of our general framework. However, within this context, our work improves and extends the results of [32] in the following ways:

1. Reconstruction is completely independent of the particular polynomial basis used. In particular, the estimates for $\Theta(n; \theta)$ and $\|f - f_{n,m}\|$ are determined only by the spaces T_n and S_m . This allows for analysis of reconstructions in arbitrary polynomial bases, not just the Legendre polynomials used in [32].
2. The estimates for $\Theta(n; \theta)$ in Theorems 3.2 and 3.3 improve those given in [32]. In particular, it was shown in [32, Theorem 4.2] that

$$C_{n,\alpha n^2} \geq 1 - \frac{8}{\pi} \arcsin \frac{1}{\pi\alpha}, \quad \forall n \in \mathbb{N}, \quad \alpha \geq 1, \quad (3.12)$$

n	5	10	15	20	25	30	35	40
(a)	3.57	5.55	4.21	5.20	4.40	5.06	4.50	6.77
(b)	13.74	49.99	52.63	91.89	92.89	133.02	133.49	191.19
(c)	3.90	5.67	7.25	9.33	11.91	13.96	16.56	18.92

Table 2: Comparison of the condition number $\kappa(A)$ with $m = 0.2n^2$, where A is formed from (a) Legendre polynomials and Chebyshev polynomials of the (b) first and (c) second kinds.

(our constant $C_{n,m}$ corresponds to the quantity $\sigma_{n,m}^2$ in [32]). Conversely, Lemma 3.1 leads to the improved bounds

$$C_{n,\alpha n^2} \geq 1 - \frac{4(\pi - 2)}{\pi^2(\alpha - n^{-2})} \quad \forall n \geq \max \left\{ \frac{2}{\pi\alpha}, \sqrt{\frac{2}{\alpha}} \right\}, \quad \alpha > 0, \quad (3.13)$$

and

$$C_{n,\alpha n^2} \geq 1 - \frac{4}{\pi^2\alpha} + \mathcal{O}(n^{-2}), \quad n \rightarrow \infty, \quad \alpha > 0. \quad (3.14)$$

Not only are these bounds sharper, they also hold for a greater range of α , thus permitting reconstruction with $m = \alpha n^2$ for any $\alpha > 0$, as opposed to just $\alpha \geq 1$. This leads to savings in computational cost, and, in cases where m is fixed, allows larger values of n to be used, thereby increasing accuracy. To illustrate this improvement, note that (3.12) gives the estimate $\kappa(A) \leq 5.71$ when $m = n^2$. Conversely, our estimate (3.13) yields the bound 2.61 for $n \geq 2$, and (3.14) gives the asymptotic bound 1.68. To compare, direct computation of $\kappa(A)$ indicates that $\kappa(A) \leq 1.32$ for all n , and $\kappa(A) \rightarrow 1.2$ as $m \rightarrow \infty$.

3. Piecewise analytic functions and function of arbitrary numbers of variables can be recovered in a analogous fashion, with similar analysis (see Sections 3.5 and 4 respectively).

3.5 Reconstruction of piecewise analytic functions

Naturally, whenever the approximated function is not analytic, the convergence rate of the polynomial approximant $f_{n,m}$ to f is not exponential. For example, consider the function

$$f(x) = \begin{cases} (2e^{2\pi(x+1)} - 1 - e^\pi)(e^\pi - 1)^{-1} & x \in [-1, -\frac{1}{2}] \\ -\sin(\frac{2\pi x}{3} + \frac{\pi}{3}) & x \in [-\frac{1}{2}, 1] \end{cases} \quad (3.15)$$

This function was put forth in [46] to test algorithms for overcoming the Gibbs phenomenon. Aside from the discontinuity, its sharp peak makes it a challenging function to reconstruct accurately. Since this function is discontinuous, we expect only low-order, algebraic convergence of $f_{n,m}$ in the L^2 norm, but no uniform convergence, an observation confirmed in Figure 6.

However, by reconstructing this function in a polynomial basis, we are not exploiting the known information about f : namely, the jump discontinuity at $x = -\frac{1}{2}$. The general procedure set out in Section 2 allows us to use such information in designing a reconstruction basis. Naturally, since f is analytic in the subintervals $[-1, -\frac{1}{2}]$ and $[-\frac{1}{2}, 1]$, a better choice is to reconstruct f in a piecewise polynomial basis. The aim of this section is to describe this procedure.

Seeking generality, suppose that $f : [-1, 1] \rightarrow \mathbb{R}$ is piecewise analytic with jump discontinuities at $-1 < x_1 < \dots < x_l < 1$. Let $x_0 = -1$ and $x_{l+1} = 1$. We assume that f has been sampled via $\hat{f}_j = \langle f, \psi_j \rangle$, $j = 1, \dots, m$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $L^2(-1, 1)$. In examples, these will be the Fourier samples of f , but the construction described below holds for arbitrary sampling bases consisting of functions defined on $[-1, 1]$.

Throughout we shall assume that the discontinuity locations x_1, \dots, x_l are known exactly. Although this may be a reasonable assumption in many applications, a fully-automated algorithm

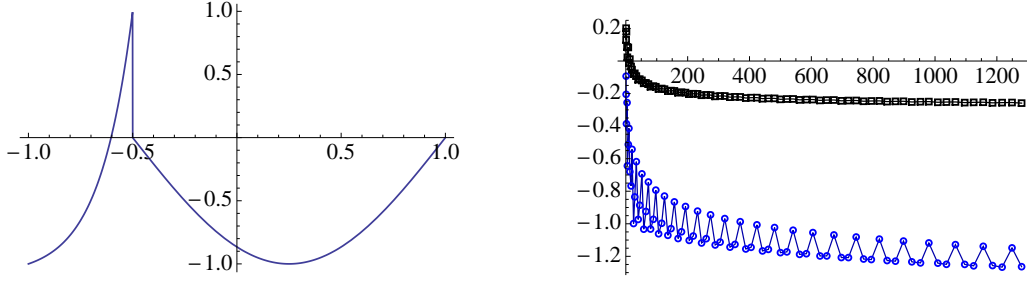


Figure 6: Error in approximating the function (3.15) by $f_{n,m}(x)$ for $n = 1, \dots, 80$. Left: the function $f(x)$. Right: log error $\log_{10} \|f - f_{n,m}\|_\infty$ (squares) and $\log_{10} \|f - f_{n,m}\|$ (circles) against $m = 0.2n^2$.

must also incorporate a scheme for singularity detection. We shall not discuss possible approaches for doing this, and we refer the reader to [45] for further details.

Given the additional information about the location of the singularities of f , we now design a reconstruction basis to mirror this feature. We shall construct such a basis via local co-ordinate mappings. To this end, let $I_r = [x_r, x_{r+1}]$, $c_r = \frac{1}{2}(x_{r+1} - x_r)$ and define $\Lambda_r(x) = \frac{x - x_r}{c_r} - 1$, so that $\Lambda(I_r) = [-1, 1]$. Suppose now that T'_n is a space of functions defined on $[-1, 1]$ (e.g. the polynomial space \mathbb{P}_{n-1}). By convention, we assume that each $\phi \in T'_n$ is extended by zero to the whole real line, i.e. $\phi(x) = 0$ for $x \in \mathbb{R} \setminus [-1, 1]$. Let $T_{n,r}$ be the space of functions defined on I_r , given by $T_{n,r} = \{\phi \circ \Lambda_r : \phi \in T'_n\}$. We now define the new reconstruction space in the obvious manner:

$$T_n = \{\phi : \phi|_{I_r} \in T_{n,r}, r = 0, \dots, l\}, \quad n = \sum_{r=0}^l n_r,$$

and seek an approximation $f_{n,m} \in T_n$ to f via the conditions $a_m(f_{n,m}, \phi) = a_m(f, \phi)$, $\forall \phi \in T_n$, where a_m is defined in (2.5). Suppose now that $\{\phi_1, \dots, \phi_n\}$ is a collection of linearly independent reconstruction functions with $T'_n = \text{span}\{\phi_1, \dots, \phi_n\}$. We construct a basis for T_n by scaling. To this end, we let $\phi_{r,j} = \frac{1}{\sqrt{c_r}} \phi_j \circ \Lambda_r$, and notice that $T_n = \text{span}\{\phi_{r,j} : j = 1, \dots, n_r, r = 0, \dots, l\}$. Note that, if $\{\phi_j\}$ are orthonormal, then so are $\{\phi_{r,j}\}$. With this basis in hand, the approximation $f_{n,m}$ is now given by

$$f_{n,m} = \sum_{r=0}^l \sum_{j=1}^{n_r} \alpha_{r,j} \phi_{r,j},$$

where the coefficients $\alpha_{r,j}$ are determined by the aforementioned equations. As before, this is equivalent to the least squares problem $U\alpha \approx \hat{f}$ with block matrix $U = [U_1, \dots, U_l]$, where U_r is the $m \times n_r$ matrix with $(j, k)^{\text{th}}$ entry

$$\langle \phi_{r,k}, \psi_j \rangle = \frac{1}{\sqrt{c_r}} \int_{x_r}^{x_{r+1}} \phi_k(\Lambda_r(x)) \psi_j(x) dx.$$

Here $\hat{f} = (\hat{f}_1, \dots, \hat{f}_m)^\top$, $\alpha = [\alpha_0, \dots, \alpha_l]$ and $\alpha_r = (\alpha_{r,1}, \dots, \alpha_{r,n_r})^\top$.

Naturally, estimation of the constant $C_{n,m}$ is vital. The following lemma aids in this task:

Lemma 3.8. *The constant $C_{n,m}$ satisfies*

$$C_{n,m} \geq 1 - \sum_{r=0}^l (1 - C_{n_r,m,r}),$$

where $C_{n_r,m,r} = \inf_{\phi \in T_{n_r,r}} \langle \mathcal{P}_m \phi, \phi \rangle$.

Proof. For $\phi \in T_n$, denote $\phi|_{I_r}$ by $\phi^{[r]}$. Assume that $\phi^{[r]}$ is extended to $[-1, 1]$ by zero, so that $\phi = \sum_{r=0}^l \phi^{[r]}$. Since $\phi^{[r]} \perp \phi^{[s]}$ for $r \neq s$, it follows that

$$1 - C_{n,m} = \sup \left\{ \frac{\sum_{r=0}^l \langle \phi^{[r]} - \mathcal{P}_m \phi^{[r]}, \phi^{[r]} \rangle}{\sum_{r=0}^l \|\phi^{[r]}\|^2} : \phi^{[r]} \in T_{n_r, r}, \ r = 0, \dots, l, \ \sum_{r=0}^l \|\phi^{[r]}\|^2 \neq 0 \right\}.$$

Note that, for $a_r \geq 0$ and $b_r > 0$, $r = 0, \dots, l$, the inequality

$$\sum_{r=0}^l a_r \leq \sum_{r=0}^l \frac{a_r}{b_r} \sum_{r=0}^l b_r,$$

holds. Setting $a_r = \langle \phi^{[r]} - \mathcal{P}_m \phi^{[r]}, \phi^{[r]} \rangle$ and $b_r = \|\phi^{[r]}\|^2$ and using this inequality gives

$$\begin{aligned} 1 - C_{n,m} &\leq \sup \left\{ \sum_{r=0}^l \frac{\langle \phi^{[r]} - \mathcal{P}_m \phi^{[r]}, \phi^{[r]} \rangle}{\|\phi^{[r]}\|^2} : \phi^{[r]} \in T_{n_r, r}, \ r = 0, \dots, l, \ \sum_{r=0}^l \|\phi^{[r]}\|^2 \neq 0 \right\} \\ &\leq \sum_{r=0}^l \sup \left\{ \frac{\langle \phi - \mathcal{P}_m \phi, \phi \rangle}{\|\phi\|^2} : \phi \in T_{n_r, r}, \ \|\phi\| \neq 0 \right\}, \end{aligned}$$

and this is precisely $\sum_{r=0}^l (1 - C_{n_r, m, r})$. \square

Let us now focus on piecewise polynomial reconstructions from Fourier samples, in which case T_n is the space of functions ϕ with $\phi|_{I_r}$ a polynomial of degree n_r . Regarding the rate of convergence of the resulting approximation $f_{n,m}$, it is a simple exercise to confirm that

$$\|f - f_{n,m}\| \leq c(\theta) c_f \sum_{r=0}^l \sqrt{n_r} \rho_r^{-n_r},$$

where $c(\theta)$ is defined in (2.23), c_f is a constant depending on f only and ρ_r is determined by the largest Bernstein ellipse (appropriately scaled) within which the function $f|_{I_r}$ is analytic. Hence, exponential convergence of $f_{n,m}$ to f . The main question remaining is that of estimating the function $\Theta(n; \theta)$ for this reconstruction procedure. For this, we have the following result, which extends Theorems 3.2 and 3.3 to this case:

Theorem 3.9. *The function $\Theta(n; \theta)$ satisfies*

$$\Theta(n; \theta) \leq 2 \left[\frac{1}{2} + \frac{2(\pi - 2)}{\pi^2(1 - \theta)} \sum_{r=0}^l \frac{n_r^2}{c_r} \right], \quad \forall n = \sum_{r=0}^l n_r, \ n_0, \dots, n_l \in \mathbb{N},$$

and

$$\Theta(n; \theta) \leq \frac{4}{\pi^2(1 - \theta)} \sum_{r=0}^l \frac{n_r^2}{c_r} + \mathcal{O}(1), \quad n_0, \dots, n_l \rightarrow \infty.$$

Proof. In view of Lemma 3.8, it suffices to consider $C_{n_r, m_r, r}$. To this end, let $J = [\alpha, \beta] \subseteq [-1, 1]$, $T_{n,J}$ be the space of functions ϕ with $\text{supp}(\phi) \subseteq J$ and $\phi|_J \in \mathbb{P}_{n-1}$, and define

$$E_{n,m} = \sup_{\substack{\phi \in T_{n,J} \\ \|\phi\|=1}} \langle \phi - \mathcal{P}_m \phi, \phi \rangle.$$

Let $\Lambda(x) = \frac{x-\alpha}{c} - 1$, where $c = \frac{1}{2}(\beta - \alpha)$, and write $\phi = \Phi \circ \Lambda$, where $\text{supp}(\Phi) \subseteq [-1, 1]$. Consider the quantity $\langle \phi, \psi_j \rangle$. By definition of ψ_j , we have

$$\langle \phi, \psi_j \rangle = \frac{1}{\sqrt{2}} \int_{-1}^1 \phi(x) e^{-ij\pi x} dx = \frac{c}{\sqrt{2}} e^{-ij\pi(\alpha+c)} \int_{\Lambda(-1)}^{\Lambda(1)} \Phi(y) e^{-ij\pi c y} dy.$$

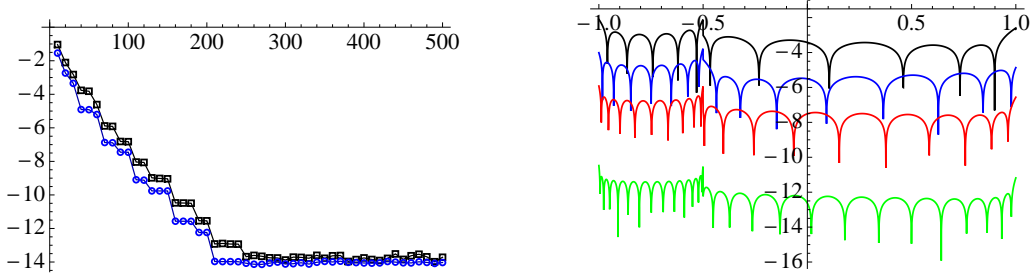


Figure 7: Error in approximating the function (3.15) by $f_{n,m}(x)$. Left: $\log_{10} \|f - f_{n,m}\|_\infty$ (squares) and $\log_{10} \|f - f_{n,m}\|$ (circles) against $m = 1, \dots, 500$ with $n_0 = n_1$ chosen so that $m = \frac{1}{5} \left(\frac{n_0^2}{c_0} + \frac{n_1^2}{c_1} \right)$. Right: the error $\log_{10} |f(x) - f_{n,m}(x)|$ against $x \in [-1, 1]$ for $m = 20, 40, 80, 160$.

Let $K = [\Lambda(-1), \Lambda(1)] = \Lambda([-1, 1]) \supseteq [-1, 1]$ and let $\mathcal{P}_{m,K}$ be the Fourier projection operator based on the interval K . It now follows that

$$E_{n,m} = \sup \{ \langle \Phi - \mathcal{P}_{m,K} \Phi, \Phi \rangle : \text{supp}(\Phi) \subseteq [-1, 1], \Phi|_{[-1,1]} \in \mathbb{P}_{n-1}, \|\Phi\| = 1 \}.$$

This is now precisely the setup of Remark 5. Using (3.6), we therefore deduce that

$$E_{n,m} \leq \frac{4(\pi - 2)n^2}{c\pi^2(2\lfloor \frac{m}{2} \rfloor - 1)}.$$

Letting $J = I_r$, $c = c_r$ and using Lemma 3.8, we now obtain

$$C_{n,m} \geq 1 - \frac{4(\pi - 2)}{\pi^2(2\lfloor \frac{m}{2} \rfloor - 1)} \sum_{r=0}^l \frac{n_r^2}{c_r}, \quad (3.16)$$

from which the result follows immediately. \square

To implement this scheme, it is necessary to compute the values (3.5). By changing variables, it is easily seen that

$$\langle \phi_{r,k}, \psi_j \rangle = \sqrt{\frac{c_r}{2}} e^{-ij\pi d_r} \int_{-1}^1 \phi_k(y) e^{-ij\pi c_r y} dy,$$

where $d_r = \frac{1}{2}(x_{r+1} + x_r)$. Since (3.4) holds for all $z \in \mathbb{C}$, it follows that

$$\langle \phi_{r,k}, \psi_j \rangle = e^{-ij\pi d_r} (-i)^k \sqrt{\frac{k + \frac{1}{2}}{j}} J_{k+\frac{1}{2}}(j\pi c_r), \quad (3.17)$$

whenever the functions $\phi_{r,k}$ arise from scaled Legendre polynomials. Naturally, if the functions $\phi_{r,k}$ arise from arbitrary scaled Gegenbauer polynomials, computation of the values (3.5) can be carried out recursively via the algorithm described in Section 3.2 (for appropriate choices of z).

In Figure 7 we apply this method to the function (3.15) using the orthonormal basis of scaled Legendre polynomials. The improvement over Figure 6 is dramatic: using only $m \approx 250$ (with $n_0 = n_1 = 16$) we obtain 13 digits of accuracy. Note that, as expected, root exponential convergence occurs. Moreover, as predicted by (3.16), and illustrated in Table 3, the condition number of the matrix A remains small.

m	10	20	40	80	160	320	640	1280
$C_{n,m}$	0.05	0.34	0.33	0.44	0.44	0.47	0.49	0.50
$\kappa(A)$	19.88	2.92	3.06	2.27	2.27	2.11	2.03	1.98

Table 3: The constant $C_{n,m}$ and the condition number $\kappa(A)$ against m .

3.6 Comparison to existing methods

Numerous methods exist for recovering functions to high accuracy from their Fourier data. Applications are myriad, and range from medical imaging [5, 6] to postprocessing of numerical solutions of hyperbolic PDEs [27, 36]. Prominent examples which deliver high global accuracy (in contrast to standard filtering techniques, which only yield high accuracy away from the singularities of a function [30]) include Gegenbauer reconstruction [29, 30, 31], techniques based on extrapolation [19], Padé methods [18] and Fourier extension methods [10, 33] (for a more complete list, see [12] and references therein).

Whilst many of these methods deliver exponential convergence in terms of m (the number of Fourier coefficients), they all suffer from ill-conditioning. This comes as no surprise: the problem of reconstructing a function from its Fourier coefficients can be viewed as a continuous analogue of the recovery of a function from m equidistant pointwise values. As proved in [42], any method for this problem that converges exponentially fast in m will suffer from exponentially poor conditioning. We conjecture that a similar result holds in the continuous case.

Aside from increased susceptibility to round-off error and noise, ill-conditioning often makes so-called inverse methods (e.g. extrapolation and Fourier extension methods) costly to implement. Conversely, the method proposed in the paper does not suffer from any ill-conditioning. This negative consequence of [42] is circumvented, precisely because we witness only root exponential convergence in m . However, an advantage of this approach is that it delivers exponential convergence in n , the degree of the final approximant $f_{n,m}$. In many applications it may be necessary manipulate $f_{n,m}$, its relatively low degree making such operations reasonably cheap. Thus, this method has the advantage of compression, a feature not shared by the majority of the other methods mentioned previously.

A well-established and widely used alternative to this method is the Gegenbauer reconstruction technique, proposed by Gottlieb et al [31]. Much like this approach, it computes a polynomial approximation. Yet it stands out as being direct, meaning that no linear system or least squares problem is required to be solved. Whilst the original method has been shown to suffer from a generalised Runge phenomenon [11], thereby severely affecting its applicability, an improved approach based on Freud polynomials was recently developed in [24].

Comparatively speaking, this approach delivers exponential convergence in $\mathcal{O}(m^2)$ operations. On the other hand, our method obtains root exponential convergence at a cost of $\mathcal{O}(m^{\frac{3}{2}})$ operations. However, despite being theoretically more efficient, the various constants involved in the Gegenbauer/Freud procedure tend to be rather large.

Indeed, in Table 4 we compare the error in approximating the function (3.15) using both procedures (the data for the Freud method is taken from [24, Table 1]. Note that the parameter N used therein is such that $2N = m$ is the total number of Fourier coefficients). As is evident, the method proposed in this paper obtains an error of order 10^{-14} using less than 256 Fourier coefficients, whereas the Freud procedure does not reach this value until more than 1024 coefficients are used.

The most likely reason for this improvement is that the generalised reconstruction method is quasi-optimal, thereby delivering a near-optimal polynomial approximation, whereas the Gegenbauer and Freud procedures do not possess this feature. Indeed, in the Gegenbauer case at least, we can quantitatively explain this phenomenon (the corresponding results for the Freud polynomial case are not so well understood). It is known that, for analytic functions f , the approximation $F_{n,m}$

m	64	128	256	512	1024	2048	4096
(a)	8.90e-01	1.37e-01	1.84e-04	1.01e-07	9.33e-13	5.27e-13	5.23e-14
(b)	2.40e-04	8.36e-09	2.40e-14	1.38e-14	1.74e-14	2.26e-14	2.59e-14

Table 4: Comparison of the (a) Freud polynomial method and (b) generalised reconstruction method applied to (3.15). Here m is the total number of Fourier samples.

obtained from the Gegenbauer procedure converges geometrically at rate γ^m , where

$$\gamma = \max \left\{ \rho^{-1}, \frac{(\beta + 2\alpha)^{\beta+2\alpha}}{(2\pi e)^\alpha \alpha^\alpha \beta^\beta} \right\},$$

(see [30, eqn. (4.12)]). Here $\lambda = \alpha m$ is the parameter of the Gegenbauer polynomials used and $n = \beta m$ is the degree of the polynomial $F_{n,m}$. Thus, in practice, the Gegenbauer method may converge much more slowly than the best polynomial approximation $\mathcal{Q}_m f$, which converges at rate ρ^{-m} . Conversely, our approach offers root exponential convergence at a rate determined solely by ρ : $\|f - f_{n,m}\| \sim \rho^{-\sqrt{m}}$.

Suppose now that the number of Fourier coefficients is fixed. Then, ignoring various constants, the Gegenbauer method will offer a lower error than the generalised reconstruction procedure only when $\gamma^m \lesssim \rho^{-\sqrt{m}}$. In other words,

$$m \gtrsim \left(\frac{\log \rho}{\log \gamma} \right)^2.$$

In typical examples (see [30, 31]), the parameter values $\alpha = \beta = \frac{1}{4}$ have been used, giving the condition $m \gtrsim 19(\log \rho)^2$. Thus, for reasonably analytic functions f (i.e. analytic in a sufficiently large Bernstein ellipse), the Gegenbauer procedure will only begin to outperform the generalised reconstruction method when the parameter m is rather large. Moreover, whenever f is entire (as is the case with the example in Table 4), the generalised reconstruction procedure will converge super-geometrically (in $n = \mathcal{O}(\sqrt{m})$), whereas the Gegenbauer method may still converge only exponentially at rate γ . Thus, in this situation the Gegenbauer method may never outperform the generalised reconstruction method for any finite m .

Aside from improved numerical performance, let us mention several other benefits. First, as discussed, the final approximation consists of only $\mathcal{O}(\sqrt{m})$ terms, as opposed to $\mathcal{O}(m)$. Second, the basis for the polynomial reconstruction space T_n can be chosen arbitrarily (in particular, independently of m) without affecting the convergence. The only downside is a mild increase in condition number if nonorthogonal polynomials are employed. In contrast, for the Freud/Gegenbauer procedure, only very specific types of polynomials can be used (which may not be simple to construct or manipulate [24]), and, whenever the number of samples m is varied, all polynomials used for reconstruction must be changed.

To somewhat temper our claims, we mention that we have only considered one particular test function. There may well be problems where the Freud/Gegenbauer procedure performs better, and the intention of future work is to present a more thorough comparison of the two methods. That aside, one advantage of Gegenbauer method is that it is local: the approximation in each subdomain is computed separately and independently of any other subdomain. Conversely, with our approach, the computations are inherently coupled. Nevertheless, it may be possible to devise a local version of our approach, a question we intend to explore in future investigations.

4 Reconstructions in tensor-product spaces

Thus far, we have focused on the reconstruction of univariate functions from Fourier samples. A simple extension of this approach, via tensor products, is to functions defined in cubes. The aim of this section is to detail this generalisation.

To formalise this idea, let us return to the general perspective of Section 2. Suppose that the Hilbert space H can be expressed as a tensor-product $H = H_1 \otimes \cdots \otimes H_d$ of Hilbert spaces H_i , $i = 1, \dots, d$, each having inner product $\langle \cdot, \cdot \rangle_i$. Note that, for $f = f_1 \otimes \cdots \otimes f_d \in H$ and $g = g_1 \otimes \cdots \otimes g_d \in H$, we have

$$\langle f, g \rangle = \prod_{i=1}^d \langle f_i, g_i \rangle_i.$$

Now suppose that the sampling basis consists of tensor-product functions. To this end, let

$$\psi_j = \psi_{1,j_1} \otimes \cdots \otimes \psi_{d,j_d}, \quad j = (j_1, \dots, j_d) \in \mathbb{N}^d,$$

and, for $m = (m_1, \dots, m_d) \in \mathbb{N}^d$, set

$$\mathcal{S}_m = \text{span} \{ \psi_j : j = (j_1, \dots, j_d), 1 \leq j_i \leq m_i, i = 1, \dots, d \}.$$

We assume throughout that the collection $\{\psi_{i,j}\}_{j=1}^\infty$ is a Riesz basis for H_i for $i = 1, \dots, d$ (hence $\{\psi_j\}$ is a Riesz basis for H). With this to hand, we define the operator $\mathcal{P}_m : H \rightarrow \mathcal{S}_m$ by

$$\mathcal{P}_m f = \sum_{j_1=1}^{m_1} \cdots \sum_{j_d=1}^{m_d} \langle f, \psi_j \rangle \psi_j.$$

Note that $\mathcal{P}_m(f_1 \otimes \cdots \otimes f_d) = \mathcal{P}_{1,m_1} f_1 \otimes \cdots \otimes \mathcal{P}_{d,m_d} f_d$, where $\mathcal{P}_{i,m_i} : H_i \rightarrow \mathcal{S}_{i,m_i}$ is defined in the obvious manner. In a similar fashion, we introduce the reconstruction vectors $\phi_j = \phi_{1,j_1} \otimes \cdots \otimes \phi_{d,j_d}$, which form a basis for the reconstruction space

$$\mathcal{T}_n = \text{span} \{ \phi_j : j = (j_1, \dots, j_d), 1 \leq j_i \leq n_i, i = 1, \dots, d \}, \quad n = (n_1, \dots, n_d) \in \mathbb{N}^d.$$

As before, we construct the approximation $f_{n,m} \in \mathcal{T}_n$ via the equations $a_m(f_{n,m}, \phi) = a_m(f, \phi)$, $\forall \phi \in \mathcal{T}_n$, where $a_m(f, g) = \langle \mathcal{P}_m f, g \rangle$, $\forall f, g \in H$.

To cast this problem in a form suitable for computations, let $U^{[i]} \in \mathbb{C}^{m_i \times n_i}$ be the matrix with $(j, k)^{\text{th}}$ entry $\langle \psi_{i,j}, \phi_{i,k} \rangle_i$. Let $U \in \mathbb{C}^{\bar{m} \times \bar{n}}$ be the matrix of the d -variate reconstruction method, where $\bar{m} = m_1 \dots m_d$ and $\bar{n} = n_1 \dots n_d$. Then, it is easily shown that

$$U = \bigotimes_{i=1}^d U^{[i]}, \quad A = \bigotimes_{i=1}^d A^{[i]},$$

where $A = U^\dagger U$, and $A^{[i]} = (U^{[i]})^\dagger U^{[i]}$, and, in this case, $B_1 \otimes B_2$ denotes the Kronecker product of matrices the B_1 and B_2 . By a trivial argument, we conclude that the number of operations required to compute $f_{n,m}$ is of order $(n_1 m_1) \dots (n_d m_d) \sqrt{\kappa(A)}$.

Recall that the spectrum of the Kronecker product matrix $B_1 \otimes B_2$ consists precisely of the pairs $\lambda \mu$, where λ is an eigenvalue of B_1 and μ is an eigenvalue of B_2 . From this, we easily deduce that

$$\kappa(A) = \prod_{i=1}^d \kappa(A^{[i]}).$$

Hence, $\kappa(A)$ is completely determined by the matrices $A^{[i]}$, with the i^{th} such matrix corresponding to the univariate reconstruction problem with sampling basis $\{\psi_{i,j}\}_{j=1}^{m_i}$ and reconstruction basis $\{\phi_{i,j}\}_{j=1}^{n_i}$. Unsurprisingly, a similar observation also holds for the quantity $C_{n,m}$:

Lemma 4.1. *The constant $C_{n,m}$ satisfies $C_{n,m} = \prod_{i=1}^d C_{i,n_i,m_i}$.*

Proof. By Lemma 2.15, $C_{n,m} = \lambda_{\min}(\tilde{A}^{-1} A)$ and $C_{i,n_i,m_i} = \lambda_{\min}((\tilde{A}^{[i]})^{-1} A^{[i]})$, $i = 1, \dots, d$, where \tilde{A} and $\tilde{A}^{[i]}$ are defined in the obvious manner. Since $\tilde{A} = \tilde{A}^{[1]} \otimes \cdots \otimes \tilde{A}^{[d]}$, the matrix \tilde{A}^{-1} is the Kronecker product of matrices $(\tilde{A}^{[i]})^{-1}$. The result now follows immediately. \square

4.1 Reconstruction of piecewise smooth functions

Having presented the general case, we now turn our attention to the reconstruction of a piecewise smooth function $f : [-1, 1]^d \rightarrow \mathbb{R}$. We shall make the significant (see later discussion) assumption that f is smooth in hyper-rectangular subregions of $[-1, 1]^d$. To this end, let $l_i \in \mathbb{N}$ for $i = 1, \dots, d$ and suppose that

$$-1 = x_{0,i} < x_{1,i} < \dots < x_{l_i,i} < x_{l_i+1,i} = 1,$$

and define $I_{r,i} = [x_{r,i}, x_{r+1,i}]$ for $r = 0, \dots, l_i$. For $r = (r_1, \dots, r_d)$, we write $I_r = I_{r_1,1} \times \dots \times I_{r_d,d}$, so that the collection

$$\{I_r : r = (r_1, \dots, r_d), r_i = 0, \dots, l_i, i = 1, \dots, d\},$$

consists of disjoint sets whose union is $[-1, 1]^d$. We assume that f is smooth within each subdomain I_r . In addition, for $r_i = 0, \dots, l_i$ and $i = 1, \dots, d$, let $c_{r_i,i} = \frac{1}{2}(x_{r_i+1,i} - x_{r_i,i})$ and set $\Lambda_{r_i,i}(x) = \frac{x - x_{r_i,i}}{c_{r_i,i}} - 1$, $x \in I_{r_i,i}$. Note that $\Lambda_{r_i,i}(I_{r_i,i}) = [-1, 1]$.

We now construct a reconstruction space. To this end, for $n \in \mathbb{N}$ let T'_n , $\dim T'_n = n$, be a space of functions $\phi : \mathbb{R} \rightarrow \mathbb{C}$ with $\text{supp}(\phi) \subseteq [-1, 1]$. Define

$$T_{n,r,i} = \{\phi \circ \Lambda_{r,i} : \phi \in T'_n\}, \quad n \in \mathbb{N}.$$

Now suppose that n is the vector (n_1, \dots, n_d) , where

$$n_i = \sum_{r=0}^{l_i} n_{r,i}, \quad i = 1, \dots, d,$$

for some $n_{r,i} \in \mathbb{N}$. We now define the reconstruction space T_n by

$$T_n = \bigotimes_{i=1}^d \bigoplus_{r=0}^{l_i} T_{n_{r,i},r,i}.$$

We seek a basis for this space. Let $\{\phi_1, \dots, \phi_n\}$, $n \in \mathbb{N}$, be a basis for T'_n , and set

$$\phi_{r,j,i} = \frac{1}{\sqrt{c_{r,i}}} \phi_j \circ \Lambda_{r,i}.$$

A basis for T_n is now given by

$$\{\phi_{r_1,j_1,1} \otimes \dots \otimes \phi_{r_d,j_d,d}, j = 1, \dots, n_{r_i,i}, r_i = 0, \dots, l_i, i = 1, \dots, d\}.$$

This framework gives a general means in which to construct reconstruction bases in the tensor-product case for functions which are piecewise smooth with discontinuities parallel to the co-ordinate axes. Suppose now that we consider the recovery of such a function from its Fourier samples. Using the above framework, we are easily able to construct a basis consisting of piecewise polynomials of several variables. The main question remaining is that of estimating the function $C_{n,m}$. However, in view of the Lemma 4.1 and the results derived in Section 3.1, a simple argument gives

Theorem 4.2. *Suppose that $n = (n_1, \dots, n_d)$, where $n_i = \sum_{r=0}^{l_i} n_{r,i}$, and let*

$$\Theta(n; \theta) = \min\{m = (m_1, \dots, m_d) : C_{n,m} \geq \theta\}, \quad \theta \in (0, 1).$$

If $\theta_1, \dots, \theta_d \in (0, 1)$ satisfy $\theta = \theta_1 \dots \theta_d$, then we may write

$$\Theta(n; \theta) = (\Theta_1(n_1; \theta_1), \dots, \Theta_d(n_d; \theta_d)),$$

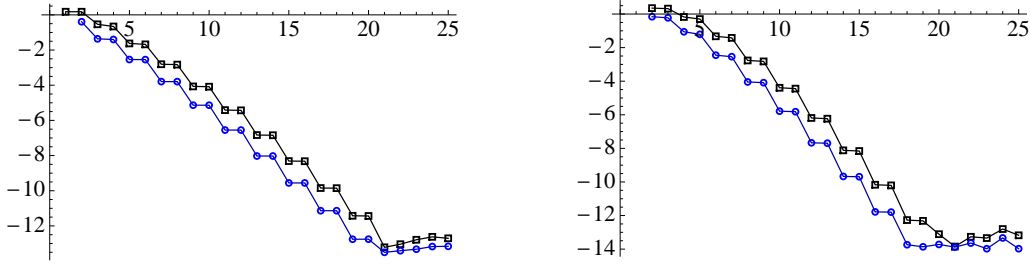


Figure 8: The errors $\log_{10} \|f - f_{n,m}\|$ (squares) and $\log_{10} \|f - f_{n,m}\|_{\infty}$ (circles) for $n_1 = n_2 = 1, \dots, 25$, where $f(x) = e^{x^2 y}$ (left) and $f(x, y) = \sin 3xy$ (right).

where, for $i = 1, \dots, d$,

$$\Theta_i(n_i; \theta_i) \leq 2 \left[\frac{1}{2} + \frac{2(\pi - 2)}{\pi^2(1 - \theta_i)} \sum_{r=0}^{l_i} \frac{n_{r,i}^2}{c_{r,i}} \right],$$

and

$$\Theta_i(n_i; \theta_i) \leq \frac{4}{\pi^2(1 - \theta_i)} \sum_{r=0}^{l_i} \frac{n_{r,i}^2}{c_{r,i}} + \mathcal{O}(1), \quad n_{0,i}, \dots, n_{l_i,i} \rightarrow \infty.$$

The main consequence of this theorem is the following: regardless of the dimension, the variables m_1, \dots, m_d must scale quadratically with n_1, \dots, n_d to ensure quasi-optimal recovery. Consider now the most simple example of this approach: namely, where f is smooth in $[-1, 1]^d$, so that T_n consists of multivariate polynomials. In Figure 8 we plot the error in approximating the functions $f(x, y) = e^{x^2 y}$ and $f(x, y) = \sin 3xy$, using parameters $m_1 = m_2 = 0.5n_1^2$ and $n_2 = n_1$. As in the univariate case, we observe the accuracy of this technique. For example, using only $m_1 = m_2 \approx 200$ and $n_1 = n_2 \approx 20$ we obtain an error of order 10^{-14} .

Remark 7 This approach (and many others based on tensor-product formulations) has the significant shortcoming that it requires the function to be singular in regions parallel to the co-ordinate axes. Naturally, this is a rather restrictive condition. For a function with singularities lying on a curve (in two dimensions, for example), one potential alternative is to apply the one-dimensional method along horizontal and vertical slices, and recover the two-dimensional function from the resulting one-dimensional reconstructions.

However, the generality of the reconstruction framework presented in this paper allows for another approach. Given that we can reconstruct in any basis, an obvious alternative is to subdivide the domain into triangular elements and use a finite element basis for reconstruction. This is a subject for future investigation.

5 Other sampling problems

Overcoming the Gibbs phenomenon in Fourier series is an obvious application of the general framework developed in Section 2. However, there is no reason to restrict to this case, and this framework can be readily applied to design effective methods for a variety of other problems. In this section we describe several related problems, and the application of this framework therein.

5.1 Modified Fourier sampling

Modified Fourier series were proposed in [34] as a minor adjustment of Fourier series. In the domain $[-1, 1]$, rather than expanding a function f in the classical Fourier basis

$$\{\cos j\pi x : j \in \mathbb{N}\} \cup \{\sin j\pi x : j \in \mathbb{N}_+\},$$

we construct the modified Fourier expansion using the basis

$$\{\cos j\pi x : j \in \mathbb{N}\} \cup \{\sin(j - \frac{1}{2})\pi x : j \in \mathbb{N}_+\},$$

instead. Though this basis arises from only a minor adjustment of the Fourier basis, the result is an improved approximation: the modified Fourier series of a smooth, nonperiodic function converges uniformly at a rate of $\mathcal{O}(m^{-1})$, whilst Fourier series suffers from the Gibbs phenomenon. Although the convergence rate remains slow, the improvement over Fourier series, whilst retaining many of their principal benefits, has given rise to a number of applications of such expansions. For a more detailed survey, we refer the reader to [4].

We shall consider modified Fourier expansions in a somewhat different context. Given the similarity between the two bases, it is reasonable to assume that any sampling procedure (e.g. an MRI scanner) can be adapted to compute the modified Fourier coefficients of a given function (or image/signal), as opposed to the standard Fourier samples. Indeed, if

$$\mathcal{F}f(t) = \int_{-1}^1 f(x)e^{-i\pi tx} dx,$$

is the Fourier transform of f , then the modified Fourier coefficients are precisely

$$\begin{aligned} \hat{f}_j^C &= \int_{-1}^1 f(x) \cos j\pi x dx = \frac{1}{2} [\mathcal{F}f(j) + \mathcal{F}f(-j)], \\ \hat{f}_j^S &= \int_{-1}^1 f(x) \sin(j - \frac{1}{2})\pi x dx = \frac{i}{2} [\mathcal{F}f(j - \frac{1}{2}) + \mathcal{F}f(\frac{1}{2} - j)], \end{aligned}$$

and hence can be computed from samples of the Fourier transform. This raises the question: given that the general framework can handle sampling in either, is there an advantage gained from sampling in the modified Fourier basis, as opposed to the Fourier basis? As we shall show, provided the function is analytic and nonperiodic, this is indeed the case. Specifically, when we reconstruct in a polynomial basis, we require fewer samples to obtain quasi-optimal recovery to within a prescribed degree.

Suppose that we carry out the reconstruction procedure as in Section 3 but using modified Fourier samples instead of Fourier samples. For this, we set

$$\mathcal{P}_m f(x) = \frac{1}{2} \hat{f}_0^C + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} [\hat{f}_j^C \cos j\pi x + \hat{f}_j^S \sin j\pi x].$$

Naturally, we consider the function $\Theta(n; \theta)$ once more. In Figure 9 we plot the function $\Theta(n; \theta)$ for the modified Fourier basis. Upon comparison with Figure 2, we conclude the following: using modified Fourier sampling, as opposed to standard Fourier sampling leads to a noticeable improvement. Specifically, $n^{-2}\Theta(n; \frac{1}{2})$ is approximately 0.15 for large n in the modified Fourier case, as opposed to 0.4 in the Fourier case.

This result means that, if the number of samples m is fixed, we are able to take a much larger value of n in the modified Fourier case, whilst retaining quasi-optimal recovery (with constant $c(\theta)$). As a result, we obtain better, guaranteed accuracy. To illustrate this improvement, in Figure 10

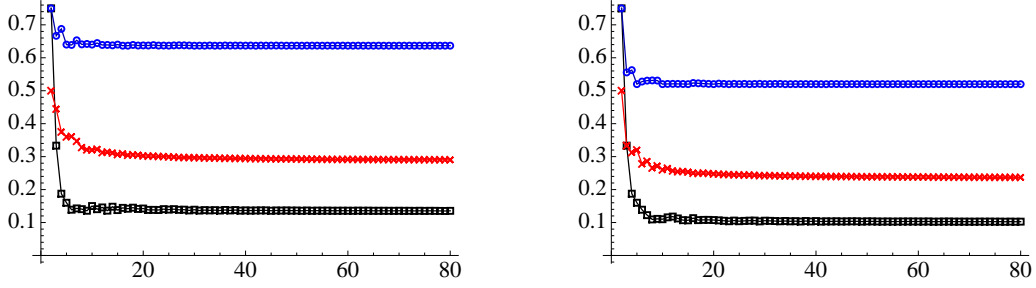


Figure 9: The function $n^{-2}\Theta(n; \theta)$ (squares), the global bound (circles) and the asymptotic bound (crosses) for $n = 1, \dots, 80$ and $\theta = \frac{1}{2}$ (left), $\theta = \frac{3}{4}$ (right).

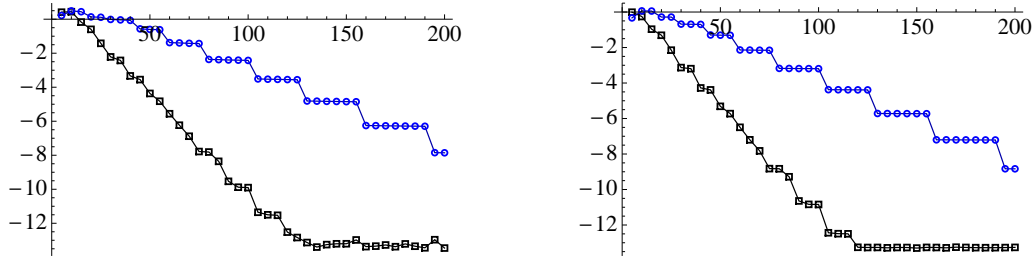


Figure 10: The errors $\log_{10} \|f - f_{n,m}\|_{\infty}$ (left) and $\log_{10} \|f - f_{n,m}\|$ (right) against $m = 5, 10, 15, \dots, 200$, where $f_{n,m}$ is computed from modified Fourier (squares) or Fourier (circles) samples.

we compare the errors in approximating the function $f(x) = e^{-x} \cos 8x$ from either its Fourier or modified Fourier data. In both cases the number of samples m was fixed, and n was chosen so that the parameter $C_{n,m} \geq \frac{1}{2}$. As is evident, the method based on modified Fourier samples greatly outperforms the other. For example, using only $m = 120$ samples, we obtain an error of order 10^{-14} for the former, in comparison to only 10^{-4} for the latter.

As in the Fourier case, to implement the modified Fourier-based approach it is necessary to have estimates for the function $\Theta(n; \theta)$. These are particularly simple to derive:

Lemma 5.1. *Let $T_n = \mathbb{P}_{n-1}$ and S_m be the space spanned by the first m modified Fourier basis functions. Then the function $\Theta(n; \theta)$ satisfies*

$$\Theta(n; \theta) \leq \left\lceil \frac{2}{\pi\sqrt{1-\theta}} k_n n^2 \right\rceil, \quad \text{where} \quad k_n = n^{-2} \sup_{\substack{\phi \in \mathbb{P}_{n-1} \\ \|\phi\|=1}} \|\phi'\|.$$

Proof. In [2] it was shown that $\|\phi - \mathcal{P}_m \phi\| \leq \frac{2}{m\pi} \|\phi'\|$ for all sufficiently smooth functions ϕ . The result now follows immediately from the definition of $C_{n,m}$. \square

As a result of this lemma, analytical bounds for $\Theta(n; \theta)$ are dependent solely on the constant k_n of the Markov inequality $\|\phi'\| \leq k_n n^2 \|\phi\|$, $\forall \phi \in \mathbb{P}_{n-1}$. Markov inequalities and their constants are well understood. The question of determining k_n was first studied by Schmidt [43], in which the estimates

$$k_n \leq \frac{1}{\sqrt{2}}, \quad \forall n, \quad \kappa_n \rightarrow \frac{1}{\pi}, \quad n \rightarrow \infty, \quad (5.1)$$

were derived. In [44] the following improved asymptotic estimate was also obtained:

$$k_n n^2 = \frac{(n + \frac{1}{2})^2}{\pi} \left[1 - \frac{\pi^2 - 3}{12(n + \frac{1}{2})^2} + \frac{R_n}{(n + \frac{1}{2})^4} \right]^{-1}, \quad n \geq 5, \quad (5.2)$$

where $-6 < R_n < 13$ (we refer the reader to [8] for a more thorough discussion of both these results and more recent work on this topic). Returning to $\Theta(n; \theta)$ we now substitute the result of Lemma 5.1 into (5.1) and (5.2) to obtain the global and asymptotic bounds. In Figure 9 we compare these bounds to their numerically computed values. The relative sharpness of such estimates is once more observed.

5.2 Polynomial sampling

The primary concern of this paper has been reconstruction from Fourier (or Fourier-like) samples. However, in several circumstances, most notably the spectral approximation of PDEs with discontinuous solutions [23, 27], the problem arises where a piecewise analytic function has been sampled in an orthogonal polynomial basis. As previously noted, this approximation will converge slowly (and suffer from a Gibbs-type phenomenon near discontinuities), hence it is necessary to seek a new approximant with faster convergence. Whilst a version of the Gegenbauer reconstruction method has been developed for this task [28, 29], the advantages of the method proposed in this paper (see Section 3.6) make it a potential alternative to this existing approach. Hence, the purpose of this section is to give a brief overview of this application.

It is beyond the scope of this paper to develop this example of the reconstruction procedure in its full generality. Instead, we consider only the recovery of a piecewise analytic function $f : [-1, 1] \rightarrow \mathbb{R}$ from its first m Legendre polynomial coefficients $\hat{f}_j = \langle f, \psi_j \rangle$, $j = 0, \dots, m-1$, where $\psi_j = (j + \frac{1}{2})^{\frac{1}{2}} P_j(x)$ is the j^{th} normalised Legendre polynomial. Proceeding as in Section 3.5, we assume that f has jump discontinuities as $-1 < x_1 < \dots < x_l < 1$, and seek an approximation of the form

$$f_{n,m}(x) = \sum_{r=0}^l \sum_{j=0}^{n_r-1} \alpha_{r,j} \phi_{r,j}(x), \quad n = \sum_{r=0}^l n_r,$$

where $\phi_{r,j}(x) = \frac{1}{\sqrt{c_r}} \phi_j(\Lambda_r(x))$, $\Lambda_r(x) = \frac{x-x_r}{c_r} - 1$, $c_r = \frac{1}{2}(x_{r+1} - x_r)$ and $\{\phi_0, \dots, \phi_{n-1}\}$ is a system of polynomials on $[-1, 1]$. Since f is piecewise analytic, we expect exponential convergence of $f_{n,m}$ to f , provided m is sufficiently large in comparison to n .

Aside from determining how large m must be in comparison to n for recovery, the main question remaining is that of implementation, i.e. how to compute the entries of the matrix U . This requires evaluation of the integrals

$$\int_{x_r}^{x_{r+1}} \psi_j(x) \phi_{r,k}(x) dx, \quad j = 0, \dots, m-1, \quad k = 0, \dots, n-1.$$

Whenever the reconstruction functions $\phi_{r,k}$ arise from Gegenbauer polynomials, these calculations can be done iteratively. For the sake of brevity, we will not describe this computation in full generality. Instead, we consider only the situation where the functions $\phi_{r,k}$ arise from Legendre polynomials, in which case we are required to compute the integrals

$$\int_{x_r}^{x_{r+1}} P_j(x) P_k(\Lambda_r(x)) dx, \quad j = 0, \dots, m-1, \quad k = 0, \dots, n-1, \quad r = 0, \dots, l.$$

We have

Lemma 5.2. *Let*

$$u_{j,k} = \int_a^b P_j(x) P_k(cx + d) dx, \quad j, k = 0, 1, 2, \dots, \quad (5.3)$$

where $ca + d = -1$ and $cb + d = 1$. Then

$$u_{0,0} = b - a, \quad u_{j,0} = \frac{1}{j} [P_{j+1}(x) - xP_j(x)]_{x=a}^b, \quad j = 1, 2, \dots,$$

$$u_{0,k} = \frac{2}{c}\delta_{0,k}, \quad u_{1,k} = \frac{2}{3c}\delta_{1,k} - \frac{2d}{c}\delta_{0,k}, \quad k = 0, 1, 2, \dots,$$

and, for $j \geq 2$ and $k \geq 1$,

$$u_{j,k} = \frac{(2j-1)(k+1)}{cj(2k+1)}u_{j-1,k+1} + \frac{(2j-1)k}{cj(2k+1)}u_{j-1,k-1} - \frac{d(2j-1)}{cj}u_{j-1,k} - \frac{j-1}{j}u_{j-2,k}. \quad (5.4)$$

Proof. Recall the recurrence relation

$$jP_j(x) = (2j-1)xP_{j-1}(x) - (j-1)P_{j-2}(x), \quad j = 2, 3, \dots, \quad (5.5)$$

for Legendre polynomials [1, chpt 22]. Substituting this into (5.3) gives

$$u_{j,k} = \frac{2j-1}{j} \int_a^b xP_{j-1}(x)P_k(cx+d) dx - \frac{j-1}{j}u_{j-2,k}.$$

Letting $x \mapsto cx + d$ in (5.5) and rearranging, we find that

$$xP_k(x) = \frac{k+1}{c(2k+1)}P_{k+1}(cx+d) - \frac{d}{c}P_k(cx+d) + \frac{k}{c(2k+1)}P_{k-1}(cx+d).$$

The recurrence (5.4) now follows upon substituting this into the previous expression.

Next consider $u_{j,0}$. Since $P_0 \equiv 1$, we have $u_{0,0} = b - a$ and $u_{j,0} = \int_a^b P_j(x) dx$ for $j \geq 1$. Recall that the j^{th} Legendre polynomial satisfies the Legendre differential equation [1, chpt 22]

$$[(1-x^2)P_j'(x)]' = -j(j+1)P_j(x).$$

Substituting for P_j in $\int_a^b P_j(x) dx$ and integrating gives

$$u_{j,0} = \frac{1}{j(j+1)} [(x^2-1)P_j'(x)]_{x=a}^b.$$

The result now follows directly from the expression

$$(1-x^2)P_j'(x) = (j+1)(xP_j(x) - P_{j+1}(x)), \quad j = 0, 1, 2, \dots$$

To complete the proof, we consider $u_{0,k}$ and $u_{1,k}$. By the assumptions on a, b, c, d , we find that

$$u_{0,k} = \frac{1}{c} \int_{-1}^1 P_k(x) dx.$$

Orthogonality now gives $u_{0,k} = \frac{2}{c}\delta_{0,k}$, as required. For $u_{1,k}$ we have

$$u_{1,k} = \frac{1}{c} \int_{-1}^1 (y-d)P_k(y) dy = \frac{1}{c} \left(\frac{2}{3}\delta_{1,k} - 2d\delta_{0,k} \right),$$

as required. □

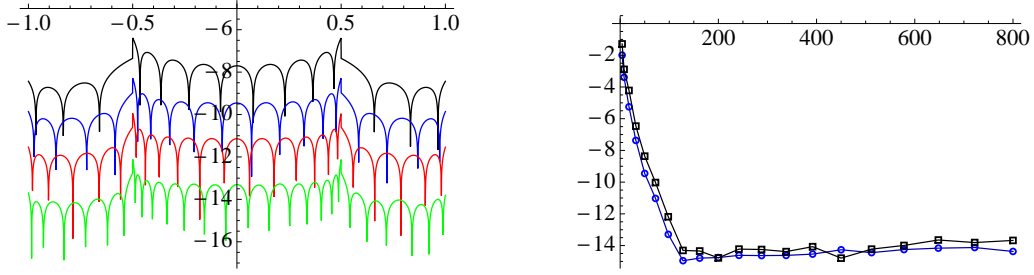


Figure 11: Left: the error $\log_{10} |f(x) - f_{n,m}(x)|$ for $-1 \leq x \leq 1$ and $m = 20, 40, 80, 160$. Right: log errors $\log_{10} \|f - f_{n,m}\|_\infty$ (squares) and $\log_{10} \|f - f_{n,m}\|$ (circles) against m .

n	8	16	24	32	40	48	56	64	72	80
$C_{n,m}$	0.98	0.87	0.85	0.85	0.85	0.84	0.84	0.84	0.84	0.84
$\kappa(U^*U)$	19.97	4.17	3.57	3.57	3.50	3.43	3.43	3.41	3.38	3.38

Table 5: The values $C_{n,m}$ and $\kappa(U^*U)$ against n , where $m = \frac{1}{8}n^2$.

In Figure 11 we consider the approximation of the function

$$f(x) = \begin{cases} \sin \cos x & -\frac{1}{2} \leq x < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \quad (5.6)$$

by the aforementioned method, using parameter values $m = \frac{1}{8}n^2$, $n_0 = n_2 = \frac{1}{4}n$ and $n_1 = \frac{1}{2}n$. As exhibited, we obtain 13 digits of accuracy using only $m \approx 120$ Legendre coefficients of (5.6). Note that, although we have not shown it, the scaling $m = \mathcal{O}(n^2)$ appears to be sufficient for recovery. Numerical results, demonstrating this hypothesis, are given in Table 5.

The function (5.6) was introduced in [28] to test the Gegenbauer reconstruction method when applied to this type of problem. As shown in Figure 11, we obtain a uniform error of roughly 10^{-8} using only $m = 40$ coefficients, and when $m = 120$, the corresponding value is 10^{-14} . In comparison, the Gegenbauer method gives errors of roughly 10^{-3} and 10^{-7} for these values of m (see [28, Fig. 3]), the latter being 10^7 times larger.

Whilst this method appears to be a promising alternative, it should be mentioned that the recursive scheme to compute the entries of U requires $\mathcal{O}(m^2)$ operations. Since only $\mathcal{O}(mn)$ operations are required to compute the approximation $f_{n,m}$, this is clearly less than ideal. Having said that, the Gegenbauer reconstruction method requires $\mathcal{O}(m^2)$ operations to compute each approximant, whereas with this scheme this higher cost is only incurred in a preprocessing step.

6 Conclusions and future work

We have presented a reconstruction procedure to recover a function using any collection of linearly independent vectors, given a finite number of its samples with respect to an arbitrary Riesz basis. This approach is both stable and quasi-optimal, provided the number of samples m is sufficiently large in comparison to the number of reconstruction vectors n . Moreover, this condition can be estimated numerically or, in certain circumstances, analytically. A prominent example of this approach is the reconstruction of a piecewise analytic function from its Fourier samples. Using a piecewise polynomial basis, this results in an approximation that converges root exponentially in terms of m , or exponentially in terms of n .

There are many potential avenues to pursue in extending this work, as we now detail:

1. *Piecewise polynomial reconstructions from polynomial samples.* In the final section in the paper we detailed the recovery of a piecewise analytic function in a piecewise polynomial basis, given its Legendre polynomial expansion coefficients. Herein, an important open problem is verifying that the scaling $m = \mathcal{O}(n^2)$ is sufficient for reconstruction. Other challenges involve devising an iterative scheme for computing the entries of U valid for reconstructions in arbitrary Gegenbauer polynomials, and which involves only $\mathcal{O}(mn)$ operations, as opposed to $\mathcal{O}(m^2)$. Naturally, future work will also investigate the extension of this approach to reconstruction from arbitrary Gegenbauer polynomial expansion coefficients (as opposed to just Legendre polynomial expansion coefficients).

2. *Gegenbauer polynomial reconstructions from Fourier samples.* As shown, the reconstruction procedure can be implemented with arbitrary Gegenbauer polynomials. However, unless Legendre polynomials are used, the reconstruction is not stable. This problem arises because Gegenbauer polynomials do not form a Riesz basis for the space $L^2(-1, 1)$ unless $\lambda = \frac{1}{2}$. However, Gegenbauer polynomials do form an orthogonal basis for the weighted space $L^2_\omega(-1, 1)$, where $\omega(x) = (1 - x^2)^{\lambda - \frac{1}{2}}$. Hence, it is natural to ask whether the reconstruction procedure can be adjusted to incorporate this additional structure, thereby yielding a stable method. It turns out that this can be done, with the first step being the derivation of an extended abstract framework along similar lines to Section 2. We are currently compiling results in this case, and will report the details in a future paper.

3. *Applications.* Aside from the obvious applications in image and signal processing, there are a number of other potential uses of the procedure. First, it may be applicable to the spectral discretisation of PDEs. Spectral methods are extremely efficient for solving problems with smooth solutions. However, for problems that develop discontinuities, e.g. hyperbolic conservation laws, a postprocessor is required to recover high accuracy [27]. The Gegenbauer reconstruction technique has recently been applied to such problems (see [23, 27] and references therein). Given the potential advantages of the method developed in this paper (see Section 3.6), it is of significant interest to apply this approach to these problems. Aside from high accuracy, a pertinent issue in the use of spectral approximations for nonsmooth problems is the question of stability [27]. Since the generalised reconstruction procedure is well-conditioned, there may also be a benefit in this regard. Outside of PDEs, the Gegenbauer reconstruction technique has also been extended to other types of series, including radial basis functions [39], Fourier–Bessel series [40] and spherical harmonics [22]. Future work will also consider generalisation of the method of this paper along these lines.

4. *Spline and wavelet-based reconstructions.* Reconstructions in spline and wavelet bases are important in numerous applications. In [3], the authors gave a first insight into the application of such bases to the Fourier sampling problem. However, the theory is far from complete. Moreover, the use of more exotic objects, such as curvelets [16] and ridgelets [15], remains to be investigated.

5. *Recovery from pointwise samples.* The discrete analogue of the Fourier coefficient recovery problem involves the reconstruction of a function from m equispaced samples in $[-1, 1]$. This problem has received more attention of late [13, 42] than the continuous case considered in this paper. In particular, the so-called mock–Chebyshev method [14] can be viewed as a discrete analogue of this approach. Whilst the mock–Chebyshev method is well understood, there remain a number of other reconstruction from pointwise samples problems of interest. In particular, with application to spectral collocation schemes based on Chebyshev or Legendre polynomials, the recovery of a piecewise analytic function from its values at Gauss or Gauss–Lobatto nodes. Future work will consider this problem.

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